Block | Hermitian Yang-Mills in derived cats

Understand stability in derived categories.

Donaldson-Uhlenbeck-Yau: Polystable vec bdl $\longleftrightarrow$ HYM.

Bridgeland: Π-stability.

Physical branes = branes satisfying diff eqn $\overset{\text{large vol lim}}{\rightsquigarrow}$ HYM.

Recollection: Given holom vec bdl (E, D'') E C^∞ bdl D'' a $\bar\partial$ cnxn

labure: $E = (C^\infty(E), t_\infty)$ (only diff. in antiholo dxn)

st $D'' \circ D'' = 0$

h is called HYM metric if

- h Hermitian on E
- Chern connection D admits λ s.t.

$$\text{Curvature} = R_D = D \circ D$$

satisfies

$$R_D \wedge \frac{\omega^{n-1}}{(n-1)!} = i\lambda \frac{\omega^n}{n!} \qquad \omega \text{ Kähler form.}$$

[nonlinear PDE, hard.]

Lubke: (E, D, h) HYM cnxn $\Longrightarrow 2r \cdot c_2(E) - (r-1)c_1(E)^2 \geq 0$

easy: use Bochner method.

Bogomolov: Same ineq true for stable bundles on surfaces. (Harder.)

②

Thm Kobayashi proved: (half of DUY thm)
Donaldson-Uhlenbeck-Yau

Thm (E, D, h) HYM over compact Kähler (X, g)

$\Downarrow$

(E, h) polystable. ie,

$(E, h) \cong \bigoplus (E_i, h_i)$

- E_i stable
- each h_i has same λ as E.

Conve: A stable holom vector bundle over compact Kähler (X, g) has unique HYM cnxn.

To study moduli for derived cats:

Construction $\mathcal{A} = (A^\bullet, d, c)$ curved diff'ntial algebra; expresses integrability prop'z.

(just as $d^2 = 0 \Leftrightarrow$ holom.)

- $A^\bullet$: pos. graded alg
- d: d deg 1, $d^2 = [c, -]$.
- c: $c \in A^2$, $dc = 0$

$\rightsquigarrow$ dg Cat ~~complexes of~~ finitely generated

object is $E = (E^\bullet, \mathbb{E})$ where
- $E^\bullet = \bigoplus_{i=-N}^{M} E^i$, each E^i fin. gen. proj. right A-mod
- $\mathbb{E}$ $\mathbb{Z}$-graded superconxn.

Here,

$$\mathbb{E} = \mathbb{E}^0 + \mathbb{E}^1 + \cdots$$

where

$$\mathbb{E} : E \otimes_{A^0} A^\bullet \longrightarrow E \otimes_{A^0} A^\bullet \quad \deg 1,$$

satisfy

- $\mathbb{E}(e\eta) = \mathbb{E}(e)\eta + (-1)^{|e|} e \cdot d\eta$
- $\mathbb{E} \circ \mathbb{E} e = -e \cdot c$ $\quad$ c like a curvature piece

$\mathbb{E}^1$ not linear, cnxn in usual sense. Other terms are linear.

Morphisms: obvious.

$\rightsquigarrow$ P_a dg cat. $\quad$ Fact: P_a Karoubi-closed, pre-Δ'd dg Cat.

Ex 1) $a'' = (A^{0,\bullet}(X), \bar\partial, 0)$ Dolbeault algebra.

$\Rightarrow$ $P_{a''}$ is dg enhance. of $D^b Coh(X)$, X compact $\mathbb{C}$ mfld.

If E holom vec bdl, no holom embedding into triv bdl. But C^∞ yes you do! And a projection!

$$E \xrightarrow{i} X \times \mathbb{C}^k \xrightarrow{j} E \quad j \circ i = p, \quad p \text{ an idempotent}, \quad p \in M_k(\mathbb{C})$$

But $\operatorname{im}(p)$ is C^∞ vec bdl; natural condition is $p\bar\partial p \bar\partial p = 0$ ie, $p \circ \bar\partial \circ p = 0$.

Note that idempotent completion of $A^{0,\bullet}(X)$ Mod wouldn't get you all this — it's smaller, just things gen. by str. sheaf.

Ex 2) X cmpct complex, $\mathcal{U} = \{U_\alpha\}$ cover by Steins.

$$\cancel{\check{C}^\bullet(\mathcal{U})}\ (C^\bullet(\mathcal{U};\mathcal{O});\ \delta) \text{ — Čech complex.}$$

opens

Then

$$P_{C^\bullet(\mathcal{U};\mathcal{O})} = \text{Toledo-Tong twisted complexes} \simeq D^b Coh(X) \text{ if } \mathcal{U} \text{ fine enough.}$$

Ex 3) Q quiver

C_Q = free cat gen by Q

$C_Q^\bullet$: fxns on nerve of C_Q.

Then

$$P_{C_Q^\bullet} = D^b Rep(Q)$$

Fix $E \in P_{\mathcal{A}^\bullet}$, X Herm cmpct. A Hermitian form on E is a Herm form on each compnt

$h = \{h_i\}$, $h_i : \bar{E}^i \otimes E^i \to A^0$

conj-linear in 1st variable

Analogue of Chern conxn:

Let $\mathcal{A} = A^\bullet X$, C-valued diff forms $\cong \oplus A^{p,q}X$, $d = \partial + \bar\partial$.

Define $*$: $\eta^* = (-1)^{\frac{1}{2}(\deg\eta+1)(\deg\eta)}\bar{\eta}$; $\eta \in A^{p,q} \Rightarrow \eta^* \in A^{q,p}$, $(\eta_1\wedge\eta_2)^* = \eta_2^*\wedge\eta_1^*$. } so η^* acts as Hilb. adjoint operator if η acts

Prop Fix $(E, \mathbb{E}'') \in \mathcal{P}_{\mathcal{A}''}$.

h Hermitian str on E. Then

$$\exists! \quad \mathbb{E}'_p : E^{\bullet} \longrightarrow E^{\bullet+p-1} \underset{A^0}{\otimes} A^{p,0}$$

$$\mathbb{E}' = \sum \mathbb{E}'_p \qquad (\deg -1)$$

(So $\quad \mathbb{E} = \mathbb{E}' + \mathbb{E}''$ $\mathbb{Z}/2$ graded super conn.)

~~Th~~ such that

- $dh(e,f) = (-1)^e \left(-h\ \mathbb{E}(e)f + he\ \mathbb{E}(f)\right)$ ← Hermitian condition.

- $(\mathbb{E})'' = \mathbb{E}''$ ie, holom proj is $\mathbb{E}''$ again.

Let $\mathfrak{g}^k := \bigoplus A^{p,q}(X, \mathrm{End}^{k+p-q})$; ~~$\mathfrak{d}$~~ $\mathbb{E}', \mathbb{E}''$ act.

If $\mathbb{E}$ Chern conn, then curvature $\bullet\ R_{\mathbb{E}} \in \mathfrak{g}^0$.

$$R_{\mathbb{E}} = \mathbb{E} \circ \mathbb{E}$$

$$\bullet\ R_{\mathbb{E}}^{*} = R_{\mathbb{E}}$$

Take supertrace: Then only terms of $R_{\mathbb{E}}$ in $A^{p,p}(X, \mathrm{End}^0 E)$ contribute.

Thm Extend Chern-Weil to $D^b_{Coh}(X)$

1) Usual charac classes
2) Deligne-Beilinson cohom
3) Bott-Chern cohom

3') Secondary Bott-Chern classes.

Gauge gps: $(E, \mathbb{E}'') \in \mathcal{P}_{A''}$.

$$GL''(\mathbb{E}) := GL^0(\mathbb{E}) \oplus A^{0,1}(X, End^{-1}E) \oplus A^{0,2}(X; End^{-2}E) \oplus \cdots$$

$$GL^0(\mathbb{E}) = \prod_i GL(E^i)$$

$$gl''(\mathbb{E}) = \bigoplus A^{0,i}(X, End^{-i}(E))$$

GL'' acts on $(E, \mathbb{E}'')$; but need unitary gp. So define

$$GL(\mathbb{E}) = GL^0(\mathbb{E}) \oplus A^{p,q}(X; End^{p-q}(E))$$

$$gl(E) = \cdots$$

Let unitary gp as

$$U(E) = \{u \in GL(E) \mid u^* u = 1\}, \quad u(E) = \{\xi \in gl(E) \mid \xi^* = -\xi\}$$

$$\text{Holom vec bdl} \xleftrightarrow{\ ?\ } \text{Vec w/ unit cnxn w/ } R\ (1,1)$$

<u>Prop</u> Fix $x \in K^0(X) = K_0(A^0(X))$

$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{K_0}(C^\infty(X;\mathbb{C}))$

Then $\exists$ correspondence

$$\left\{ \begin{array}{l} (E, \mathbb{E}'') \in \mathcal{P}_{\mathcal{A}''} \text{ s.t.} \\ [E] = x \quad \underset{\text{s.t.}}{(\cancel{\text{mod } GL''(E)})} \end{array} \right\} \Big/ GL''(E)$$

$$\left\{ \begin{array}{l} \text{unitary conn} \\ \mathbb{E} : E^\bullet \to E^\bullet \otimes_{A^0} A^{p,q} \\ \mathbb{E}^{p,q} : E^\bullet \to E^{\bullet+p-q+1} \otimes A^{p,q+1} \\ \qquad\qquad\quad \overset{+}{E^{\bullet-p+q-1}} \otimes A^{p+1,q} \\ \text{s.t. } R_{\mathbb{E}} \in \mathfrak{g}^0 \ \cancel{\text{mod } U(E,h)} \end{array} \right\} \Big/ U(E,h).$$

Note $GL(E)$ takes you <u>out</u> of $\mathcal{P}_{\mathcal{A}''}$; gives new object

$$(E, \mathbb{E}''_{p,q}) \quad E^\bullet \to E^{\bullet+p-q+1} \otimes A^{p,q+1}, \ \exists \text{ flat conn}$$

enlarging cat. to include these, get bigger dg cat $\mathcal{P}_{\mathcal{A}'''}$.

This is dg-enhancement of Coh shvs on ringed space (X, Ω^-_X) — grading of forms in negative degrees, no differential

$$\cong T_X[-1].$$

This is CY of dimension zero.

The projection $T_X[-1] \underset{s}{\overset{p}{\rightleftarrows}} X$

$$p^*: D^b_{coh}(X) \longrightarrow D^b_{coh}(T_X[-1])$$

is faithful + conservative.

Natural outcome of enlarging gauge group.