A. Bondal

j/w A. Bodzenta

$X = \mathbb{P}^1/_k$ char $k = 0$ $(k = \mathbb{C})$

$D(\mathbb{P}^1) \xrightarrow{\sim} D(\cdot \rightrightarrows \cdot)$
$A \longmapsto R\mathrm{Hom}(E, A)$

$E = \mathcal{O} \oplus \mathcal{O}(1)$ $\langle E \rangle = D(\mathbb{P}^1)$

$\mathrm{Ext}^{>0}(E, E) = 0$

$A = \mathrm{End}(E)$

$D(\mathcal{A}) \xrightarrow{\sim} D(\mathcal{B})$ $\mathcal{A}, \mathcal{B}$ ab. cats. 2 dif. t-structures in the same cat.

D triangulated

Def: $D \supset (D^{\leq 0}, D^{\geq 1})$

1. $\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$ $\Rightarrow D^{\geq 1} = (D^{\leq 0})^{\perp}$
2. $D^{\leq 0}[n] \subset D^{\leq 0}$ $n \geq 0$
3. $D^{\geq 1}[n] \subset D^{\geq 1}$ $n \leq 0$
4. $\forall A \in D$ $\exists$ $\underset{D^{\leq 0}}{\underset{\cap}{A_{\leq 0}}} \to A \to \underset{D^{\geq 1}}{\underset{\cap}{A_{\geq 1}}} \to$ triangle

$\mathcal{C} = D^{\leq 0} \cap D^{\geq 1}[1]$ heart $\rightsquigarrow$ ab. cat.

Example: $D = D(\mathcal{A})$

$D^{\leq 0} = D^{\leq 0}(\mathcal{A})$, $D^{\geq 1} = D^{\geq 1}(\mathcal{A}) \Rightarrow$ $\mathcal{C} = \mathcal{A}$

$f: X \longrightarrow Y$ X, Y alg. var.

- $Rf_* \mathcal{O}_X = \mathcal{O}_Y$
- $\dim f \leq 1$
- f proper

$D(X)$
$\cup$
$\mathcal{C}_f = \langle A \in D(X) \,/\, Rf_* A = 0 \rangle$
null cat.

$(\mathcal{C}_f^{\leq 0}, \mathcal{C}_f^{\geq 1})$

restriction of the standard t-structure

(BBD)

$$\begin{tikzcd} A \arrow[r, hook, "i_*"] & D \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \end{tikzcd} \quad \text{admissible} \quad (\exists\, i^!, i^*)$$

$$\begin{tikzcd} A \arrow[r, hook, "i_*"] & D \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \arrow[r, "q"] & \mathcal{B} \end{tikzcd}$$

$$\begin{array}{ccc} \cup & & \cup \\ (A^{\le 0}, A^{\ge 1}) & & (\mathcal{B}^{\le 0}, \mathcal{B}^{\ge 1}) \end{array}$$

$\exists!\ (\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ s.t. i_* and q are t-exact

In the case above:

stand. t-str.

$\mathcal{C}_f \longrightarrow D(X) \longrightarrow D(Y)$

restric. t-str.

You get a t-structure in $D(X)$ which is not the standard one

$D(X) := D^b_{coh}(X)$

$p \in \mathbb{Z}$

${}^pD^{\le 0} = \{ A \in D(X) \mid Rf_*(A) \subset D(Y)^{\le 0},\ \mathrm{Hom}(A, C) = 0\ \forall C \in \mathcal{C}_f^{>p} \}$

(In our case above $f : \mathbb{P}^1 \to *$)

$p = -1, 0$

${}^p\mathrm{Coh}(X)$

under t.f. conditions:

- $Y \cong \mathrm{Spec}\, R$ (*)

or

- f proj.
- X, Y quasi proj.
- Exc ...

Consider case (*):

$\exists\ \mathcal{M} \in {}^{-1}\mathrm{Coh}(X)$ $\quad \mathcal{M}$ a proj. generator for ${}^{-1}\mathrm{Coh}(X)$ (vect. bundle on X)

$N = \mathcal{M}^*$ " " for ${}^{0}\mathrm{Coh}(X)$

$\Rightarrow\ {}^{-1,0}\mathrm{Coh}(X) \simeq A_p\text{-mod} \qquad A_p = \mathrm{End}\, P \qquad P = \mathcal{M}, N$

Goal: Find a t-structure with a canonical local injective generator

$X \xrightarrow{f} Y$ ~~with the same conditions as above~~

- f proj.
- f birational
- X, Y smooth
- $\dim f \leq 1$

Thm: (Danilov)

$f : X \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots \longrightarrow Y$

blow-ups with smooth centers of codim 2

exc. locus

$$\mathrm{Conn}\, f = \left\{ g \,/\, \begin{tikzcd} X \arrow[dr, "g"] \arrow[d, "f"'] & \\ Y & Z \arrow[l, "h"] \end{tikzcd} ,\ Z \text{ smooth},\ Ex(g) \text{ connected} \right\} \text{ poset}$$

$\wr\!\wr$

$Irr(Ex(f)) \longleftarrow$ poset

Def: Relative tilting object w.r.t. f $\quad A \in D(X)$ s.t.

- $Rf_* R\mathrm{End}(A) \in \mathrm{Coh}(Y)$
- $\exists\, U_i \subset Y$ open $\quad \bigcup U_i = Y$

$D(f^{-1}(U_i)) \simeq \langle A|_{f^{-1}(U_i)} \rangle_{\oplus}$

$$\begin{tikzcd} f^{-1}(U) \arrow[r, hook] & X \arrow[d, "f"] \\ U_i \arrow[r, hook] & Y \end{tikzcd}$$

$\mathcal{A}_A \underset{\text{qis.}}{\simeq} Rf_* R\mathrm{End}(A)$

sheaf of alg./Y

Thm: $D(X) \simeq D(\mathcal{A}_A\text{-mod})$

Thm: under the conditions above:

$A \in D_X$ is an f-relative tilting generator

$A = \omega_f \oplus \big(\bigoplus_{g \in \mathrm{Conn} f} \omega_f|_{D_g} \big)$

$0 \longrightarrow \omega_g \longrightarrow \omega_g|_{D_g}$

$\mathcal{O}(D_g)$