Antwerp

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joint with Iyama, Yamaura

What are the matrix factorizations of $y^d - x^d$?

$k[x]/(x^d)$ is of finite representation type, indecomposables are $M_i = k[x]/(x^i)$ Dynkin

↪ $y^d - x^d$ should correspond to endomorphisms of MCM i.e.

$$\mathrm{End}\left(\underline{\mathrm{MCM}}\ k[x]/(x^d)\right) \overset{?}{\simeq} \mathrm{MF}(y^d - x^d)$$ non-Dynkin

$d \geq 5$: $y^d - x^d$ is of ~~finite~~ wild representation type..

1 Gorenstein singularities and Orlov's theorem

$A = \bigoplus_{i\geq 0}^{\infty} A_i$, $A_0 = k$ field, A noetherian   < implies equality

A Gorenstein if a) $\mathrm{injdim}_A A = \mathrm{injdim}\ A_A < +\infty$

b) $\mathbf{R}\mathrm{Hom}_A(k, A) \simeq k(a)[d]$

$$\Leftrightarrow \quad \mathrm{Ext}^i_A(k, A) = \begin{cases} 0 & i \neq d \\ k(a) & i = d \end{cases}$$

in the commutative setting: b) implies a)

no idea in the noncommutative setting

Orlov's theorem (2005)   strongest result since Hilbert's syzygy theorem

Assume $A = \bigoplus A_i$ graded Gorenstein

and choose a cutoff $i \in \mathbb{Z}$

M is MCM if $\mathrm{Ext}^i_A(M, A) = 0$ if $i \neq 0$

usually no adjoint on this small level e.g. for a [illegible]!

$$\begin{tikzcd}
& D^b(\mathrm{mod}^{\mathbb{Z}} A) \arrow[dl, shift left, "a"] \arrow[dr, shift left, "\mathrm{res}_M"] \arrow[dr, bend right, "b"'] & \\
D^b(\mathrm{qgr}\, A) \arrow[ur, shift left, "\mathbf{R}\Gamma_{\geq i}"] \arrow[rr, shift left, "\Phi_i = \pi_0 \mathbf{R}\Gamma_{\geq i}"] & & D^b_{sg}(A) \simeq \underline{\mathrm{MCM}}(A) \arrow[ll, shift left, "\Psi_i = a \circ b"] \arrow[ul, shift left]
\end{tikzcd}$$

i) $\Phi_i \dashv \Psi_i$

ii) depending on the sign of a: $\begin{cases} \Psi_i \text{ fully faithful} & a < 0 \text{ Fano} \\ \text{equivalence} & a = 0 \text{ Calabi-Yau} \\ \Phi_i \text{ fully faithful} & a > 0 \text{ general type} \end{cases}$

and the complement is explicitly known

In the Fano case: often (tilting objects!)

2 The curve case $d = 1$

A commutative for now (orders are probably good too), denoted R

$Q = N^{-1}R$ localisation at all homogeneous nonzero divisors

then a can be read off as

$a(R) = \max \{ i \mid R_i \to Q_i \text{ is not an iso} \}$

Lemma: $\dim R = 1$, R reduced: $a < 0 \iff R = k[x]$, $|x| > 0$

Proof: $Q_0 : k$ is an extension, Q_0 reduced

$\Rightarrow Q_0$ product of fields $\prod_{i=1}^{l} k_i$, k_i / k finite

if $a < 0 \Rightarrow$ already $=$

$\Rightarrow Q_0 = k$

$\Rightarrow Q = k[t, t^{-1}]$, $\deg t > 0$

$\Rightarrow R = k[t]$, $|t| > 0$

$\square$

no interesting NCR! not necessary

Assume R reduced, $a > 0$, $\gcd \{ \deg r \mid r \in R \text{ homogeneous non-zero divisor} \} = 1$

Counterexample

$k[x,y]/(y^2)$: $a = \deg y - \deg x$

$\rightsquigarrow$ any a-invariant is possible

k_0 is too small for tilting!

in Orlov: qgr A is trivial to describe for $d = 1$!

<u>Theorem</u> R as above, consider

$M_i = R_{\geq i}(i) \qquad i = 1, \dots, a$

$N_j =$ factors of $Q_{>0} \cong k^1(F)$, $|H| > 0$, $j = 1, \dots, r$

1) $K_0(\underline{\mathrm{MCM}}^{\mathbb{Z}} R) \cong \mathbb{Z}^{a+r}$

more precisely

$$\bigoplus_{i=1}^{a} M_i \oplus \bigoplus_{j=1}^{r} N_j =: T$$

is a tilting object

2) In particular $\underline{\mathrm{MCM}}^{\mathbb{Z}} R \cong D^b(\Lambda)$, $\Lambda = \underline{\mathrm{End}}_R(T) \overset{!}{=} \mathrm{End}_R(T)$

$$\Lambda = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} M_1 & - & M_a \end{array} & \begin{array}{ccc} N_1 & - & N_r \end{array} \\ \left[\begin{array}{ccc|ccc} k & & 0 & & & \\ & \ddots & & & 0 & \\ R_{i-j} & & k & & & \\ \hline k & - & k & k & & 0 \\ | & & | & & \ddots & \\ k & - & k & 0 & & k \end{array}\right] & \end{array} \begin{array}{c} \\ M_1 \\ | \\ M_a \\ N_1 \\ | \\ N_r \end{array}$$

Λ artinian, gldim $\Lambda < \infty$

The quiver of Λ is directed $\Rightarrow$ fin gldim

is exact value?

<u>Examples</u>

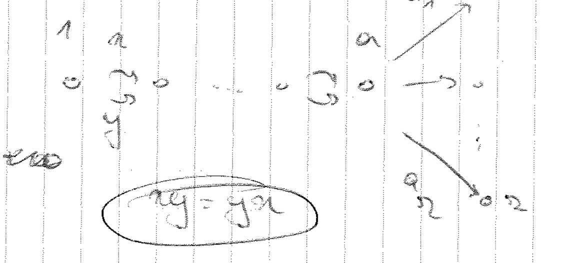
(1) line arrangements $k[x,y]/(f_1 \cdots f_d)$, f_i linear, $(f_i) \neq (f_j)$

$\hookrightarrow$ d lines

$a = d-2$, $r = d$, $d \geq 2$ here

$K_0(\underline{\mathrm{MCM}}\, R) \cong \mathbb{Z}^{2d-2}$

The quiver:



two

$xy = yx$

gldim $\Lambda = 2$

$a_i f_j = 0$

ii) symmetric numerical semigroups: $S \subseteq \mathbb{N}_{\geq 0}$ if

a) $0 \in S$

b) $a, b \in S \Rightarrow a+b \in S$ submonoid

c) $\# \mathbb{N} \setminus S < \infty$ finite numerical

$=: g$ genus of S Weierstrass points of Riemann surfaces!

symmetric if

d) $x \in \mathbb{Z}$: $x \in S$ or $2g-1-x \in S$ $\Leftarrow$ $2g-1 \notin S$

$\Rightarrow$ max $\# \mathbb{N} \setminus S \leq 2g-1$, ~~symmetric~~ $(=)$?

$k[S] = k[t^n \mid n \in S]$ Gorenstein $\Leftrightarrow$ S symmetric

$a = 2g-1$

$r = 1$! $\dim R_{-1} \leq 1$ tic tac toe algebra

~~the case~~

n $y^d - x^d$, $f = f_1 \cdots f_d$, $k = \bar{k}$, char $k >> 0$

hypersurface, so

$\underline{MCM}$ $k[x,y]/(x^d - y^d) \simeq \underline{MF}^{\mathbb{Z}}(y^d - x^d)$

$\downarrow$

what physicists wanted

$M_i = R_{\geq i}(i)$ $1 \leq i \leq d-2$

$0 \to R_{\geq i} \to R \to R/R_{\geq i} \to 0$

$0 \to S^i \xrightarrow{\begin{pmatrix} x & y & & \\ & \ddots & \ddots & \\ & & x & y \end{pmatrix}} S^{i+1} \xrightarrow{(x^i, x^{i-1}y, \dots, y^i)} S \to k[x,y]/(x,y)^i \to 0$ in three degrees

Hilbert

$S = k[x,y]$