Calaque | Derived sympl + Poisson geom

Motivation G reductive Σ oriented topological surface.

$\mathrm{Loc}_G(\Sigma)^{sm}$ ← smth part of moduli of G-local systems

has symplectic structure. Why?

$$T_{triv} \mathrm{Loc} \cong H^1(\Sigma, g)$$

Pairing: via Cartan killing form (any non-deg 2-form)

$$H^1(\Sigma, g)^{\otimes 2} \longrightarrow H^2(\Sigma, \mathbb{C}) \cong \mathbb{C}.$$

Closedness is hard.

If $\Sigma = \partial M$, M 3-mfld, have

$$\text{"} \mathrm{Loc}_G(M) \subset \mathrm{Loc}_G(\Sigma)^{sm} \text{"}$$

really, all the loc sys on Σ that extend to M. This is Lagr.

Likewise, replace $\Sigma \longleftrightarrow$ K3 surface
$\mathrm{Loc}_G \longleftrightarrow \mathrm{Bun}_G$.

Then $\mathrm{Bun}_G(K3)^{smth}$ symplectic, (Mukai)
If $K3 \hookrightarrow M$ Fano 3-fold, $[K3] = [\omega^{-1}]$
have "$\mathrm{Bun}_G(M) \subset \mathrm{Bun}_G(K3)$" is Lagrangian. (Tyurin)

Really, need cohom. constraints on H^* of G-bundles you consider.

One goal: · Set up a general framework for such results

- get rid of " " quotation marks
  (involves saying something abt non-smooth locus)
- quantize moduli spaces

If $[P]$ is non-smooth part of $\mathrm{Bun}_G(K3)$, then

$$T_{[P]}\mathrm{Bun}_G(K3) \simeq \underbrace{H^0(K3, \mathrm{ad}\,P)}_{\substack{-1 \\ \text{isotropy/} \\ \text{autom gp} \\ \text{of } P}} \oplus \underbrace{H^1(K3, \mathrm{ad}\,P)}_{0} \quad \text{cohom degree}$$

So need something to pair against the deg -1 "H^0" part — incorporate H^2 in deg 1.
This involves derived enhancement

$$\mathbb{B}\mathrm{un}_G(K3)$$

st

$$T_{[P]}\mathbb{B}\mathrm{un}_G(K3) \simeq H^{*+1}(K3, \mathrm{ad}\,P)$$

That's the point of derived geometry.

Diff geom on cdgas : $A \in cdga_{\leq 0}$ rings have negatively graded bits, differential.

Any computation for stacks boils down to cdga's. (Affines.)

Assume A cofibrant-in-fact, quasi-free, so

$$A \simeq (\mathrm{Sym}\, V^\bullet, d). \qquad \text{(Everything char } 0)$$

DeRham "algebra":

$$DR(A) := \mathrm{Sym}_A(\Omega^1_A[-1]) \quad \text{a graded mixed complex.}$$

- auxiliary grading called "weight"

ie, Sym^d, d = weight.

- another differential commuting w/ d_A,

ε = universal derivation, d_{dR}

of degree +1 cohomology.

~~degree~~ +1 weight.

ie, deg 1, wt 1.

- $d_A =: d_{int}$ has deg 0, wt 0.

GrMix Cmplx = $Cpx^{\varepsilon\text{-}gr}$ has model structure, just transferred from Cpx^{gr}.

$Cpx^{\varepsilon\text{-}gr} \xrightarrow{\text{forget } \varepsilon} Cpx^{gr}$ $\rightsquigarrow^{\text{induces}}$ model structure.

Recipe for new symp stacks. Since symp $\Rightarrow$ Pois, get quantization.

$\underline{\underline{x}}$

- $\mathcal{O}$-compact derived stack.
- d-oriented

n-symp stack

$\Bigg\} \Longrightarrow \exists$ $(n-d)$-symp on $\mathrm{Map}(X, Y)$

me $\mathrm{Map}(X,Y)$ Artin, loc fin pres stack

[in cat of <u>derived</u> stacks, even if X, Y NOT derived.

wo ex. of $\mathcal{O}$-compact stacks: $\bullet$ X smth proj d-CY

$\bullet$ $X = \Sigma_B$ ← Betti ~~loc syst,~~ Σ d-dim oriented top mfld

um $Y = BG \Rightarrow$ $\mathrm{Map}(X, BG) = \mathrm{Bun}_G(X)$ $(2-d)$-symp

ex: $d=2$, X 2-CY, ex K3, then $\mathrm{Bun}_G(X)$ is 0-symp, so symp on smth part

and $\mathrm{Map}(\Sigma_B, BG) = \mathbb{L}\mathrm{oc}_G(\Sigma)$.

Compatible w/ boundaries:

Y n-symp. stack. Thm $\mathrm{Map}((-)_B, Y)$ is a TFT w/ values in Lagrangian correspondences Lag_{n-d+1}.

(d-1) or. closed mn $\longmapsto$ (n-d+1) symp.

cob $\longmapsto$ Lag corr

Quantize Y, and then the TFT.

Strategy for $\mathrm{Loc}_G(\Sigma)$: 1) Quantize BG $\mathrm{Rep}(G)$ so quantum groups is $\mathbb{E}_2$-def; $\mathrm{Rep}(U_q\mathfrak{g})$. / n-symp Y $=$ 1st ord def of $\mathrm{Shv}(Y)$ as $\mathbb{E}_n$-cat

$\mathrm{Loc}_G(\Sigma) = \mathrm{Map}(\Sigma_B, BG)$

2) Quantize $\mathrm{Map}(-, BG)$ TFT_1 via factorization homology.

One case worked out: Ben-Zvi, Brochier, Jordan Compute

$$\int_\Sigma \mathrm{Rep}(U_q G)$$

and prove: • gives back quantization, in above case, of character varieties.

• $\int_\Sigma$ are Rep(some alg), the algebra is alg of observables in combinatorial quantization of Chern-Simons thy. (Alekseev)