Dyckerhoff | Secondary Dold-Kan

⓪ "Pre-primary" integers

① "Primary" ab gps (Dold-Kan as usual)

② "Secondary" dg cats
or stable ∞cat.

categorify

jt w/ Kapranov
Schechtman
Soibelman
Jasso (analytic case)
Poguntke

⓪ χ euler char
① $H_0(; A)$ homlgy.
② 2ndary homlgy
w/ coeff in $\mathcal{C}$,
comparing Fukaya in top. setting.
"Borel-Moore" ⟷ wrapped Fukaya.

⓪ Preprimary: $a = (a_0, a_1, a_2, \dots)$ sequence of integers "discrete fcn"

⇝ sequence of "forward differences"

$$\Delta a = (a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots)$$ "discrete derivative"

iterate ⇝ Newton's formula $$a_n = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k a)_0 .$$ "discrete Taylor's formula"

$$f(x) = \sum \frac{x^k}{k!} f^{(k)}(0)$$

Organize to a correspondence:

$$\begin{Bmatrix}\text{Integer}\\ \text{sequences}\end{Bmatrix} \xleftrightarrow{\ \cong\ } \begin{Bmatrix}\text{integer}\\ \text{sequences}\end{Bmatrix}$$

$$(a_n) \xmapsto[\text{forward diff}]{} (a_0, (\Delta a)_0, (\Delta^2 a)_0, \cdots)$$

$$\xleftarrow[\text{Newton's formula}]{} (b_0, b_1, b_2, \cdots)$$

Ex Computing $\Delta^n a$ can tell you whether (a_i) is polynomial: ie,

" $a_n = P(n)$ polyn of deg $\leq k$ "

$$\Updownarrow$$

$(\Delta^m(a))_0$ terminates for $m > k$.

$$\parallel$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{m-i}$$

Ex $(0, \cdots, 0, \underset{k}{1}, 0, \cdots) \underset{\text{Newton's formula}}{\rightsquigarrow} \binom{n}{k}$

1 Classical Dold-Kan equivalence of cats

$$C : SAb \longleftrightarrow Ch_{\geq}(Ab) : N$$

$\quad\;\; \| $
$\quad Ab_\Delta$
smp ab gp

where

$$C(A_\cdot) = \quad A_n / \text{deg.}$$ "normalized chains"

$$d = \sum_{i=0}^{n} (-1)^i d_i$$

N a nerve construction: two steps

(1) universal cosimplicial object

$$C^\cdot : \Delta \longrightarrow Ch_{\geq 0}(Ab)$$
$$[n] \longmapsto C(\mathbb{Z}\Delta^n)$$

(2)

$$N(B_\cdot) := \underline{\mathrm{Hom}}(C^\cdot, B_\bullet) \in Ab_\Delta$$

Rmk Ab_Δ presheaf cat, so gen by colims from $\mathbb{Z}\Delta^n$, so density right adj is enough

Ex $C^0 = \mathbb{Z}$

$C^1 = \mathbb{Z} \oplus \mathbb{Z} \longleftarrow \mathbb{Z}$

$C^2 = \mathbb{Z}^3 \longleftarrow \mathbb{Z}^2 \longleftarrow \mathbb{Z}$

$\vdots$

$$N(B_\cdot) = B_0 \leftleftarrows \left\{ \begin{matrix} b \in B_1 \\ b_0, b_1 \in B_0 \\ db = b_1 - b_0 \end{matrix} \right\} \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \left\{ \quad \right\}$$

$\cong B_1 \oplus B_0 \qquad B_2 \oplus B_1^2 \oplus B_0$

General formula:

$$N(B_\bullet)_n \cong \bigoplus_{\substack{[n] \twoheadrightarrow [k] \\ \text{surj.}}} B_k$$

collecting back degenerate cells

Pass to ranks: Get Newton's formula:

$$\sum_{k=0}^{n} \binom{n}{k} b_k .$$

ie,

$$\begin{tikzcd} Ab_\Delta \arrow[r, shift left, "\text{Dold-Kan}"] \arrow[d, "\text{rks}"'] & Ch_{\geq 0}(Ab) \arrow[l, shift left] \arrow[d, "\text{rks}"] \\ \{(a_n)\} \arrow[r, shift left] & \{(b_n)\} \arrow[l, shift left, "\text{Newton}"] \end{tikzcd}$$

(observed by Joyal)

Feature, $|N(\mathbf{B}[n])| \simeq K(\mathbf{B}, n).$

$$\binom{n}{k} \mapsfrom (0 \cdots 0, 1, 0 \cdots 0, \ldots)$$

② Secondary: dg cats up to Morita equiv. (5)

Note "$\sum (-1)^i d_i$" doesn't make sense, since can't take differences of functors

dg cats are semiadditive, not additive.

Fact $\exists$ not quotient of A_n, but subobject of A_n construction of DK. (of C).

The alternative cstrxn:

$$Ab_\Delta \ni A_\bullet \longmapsto C(A_\bullet)_n = \bigcap_{\substack{i=1 \\ (\text{NOT } i=0)}}^{k} \ker(d_i)$$

makes sense for dg cats.

$$d = d_0.$$

Thm $\bigcap \ker \hookrightarrow A \twoheadrightarrow A/\deg$ composition is an equivalence!

Ex $C^1 = \mathbb{Z} \oplus \mathbb{Z} \leftarrow \mathbb{Z}$

$\{0,1\}$ $\{01 - 00\}$

$C^2 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}$

$\{0,1,2\}$ $\begin{matrix}(01-00)\\(02-00)\\(12-11)\end{matrix}$ $(012 - 011 - 002 - 001)$

(6)

For $A_\bullet \in (\text{dg}\,\text{Cat})_\Delta$ ie, thems $N(\Delta^{op}) \to \text{dg Cat}$
∞ cat,
localized @ Mor??

let

$$C(A_\bullet) := \bigcap_{i=1}^{n} \text{Ker}(d_i)$$

makes sense to talk of kernel.

perfect envelope

$$\begin{tikzcd} \mathcal{C} \arrow[d] & \sim & \text{Ker} \arrow[r] \arrow[d] & \text{Perf}(\mathcal{C}) \arrow[d] \\ \mathcal{D} & & \ast \arrow[r] & \mathcal{D} \end{tikzcd}$$

$d = d_0$

$\rightsquigarrow$ "chain complex of dg cat"

$$Ch_{\geq 0}(\text{dg Cat})$$

$C(A_\bullet) =$ "secretly normalized chains"

Secretly nerve: $\mathcal{C}^\bullet : \Delta \longrightarrow Ch_{\geq 0}(\text{dg Cat})$

$$[n] \longmapsto \underbrace{\mathbb{Z}\,\text{Hom}(-, [n])}_{\text{Cat}_\Delta}$$

Define $N(B_\bullet) := \underline{\text{Hom}}_{\text{Cat}_\Delta}(\mathcal{C}^\bullet, B_\bullet)$.

Ex

$$\mathcal{C}^1 = \mathbb{Z}\{0 \to 1\} \longleftarrow \mathbb{Z}\{\overset{\text{pairs}}{\underbrace{01-00}_{\text{"take cone of"}}}\}$$

= ln cat gnby ths = A_2-Mod. $\quad\quad$ $\mathbb{Z}\{01-00\} = A_1$Mod

$$\underset{\text{Cone}(0\to 1)}{1-0} \xleftarrow{\text{do}} (01-00)$$

$$\mathcal{C}^2 = \mathbb{Z}\{0 \to 1 \to 2\} \xleftarrow{\text{do}} \mathbb{Z}\left(\begin{array}{ccc} (01-00) & \to & (02-00) \\ & \swarrow & \downarrow \\ 0 & \to & (12-11) \end{array}\right) \xleftarrow{\text{do}} \mathbb{Z}\left\{\begin{array}{c} (012-011) \\ -(002-001) \end{array}\right\}$$

$\mathcal{C}^2 = A_3$-Mod $\quad\quad$ $\cong A_3$Mod $\quad\quad$ $\longleftarrow$ totalization of 2-term complex.

$K_0 \rightsquigarrow$ unusual examples for below.

In gnl: $$(\mathcal{C}^n)_{k-1} = \mathbb{Z}\left\{\begin{array}{c}\text{pairs of } (k-1)\text{simplices} \\ \text{in } \Delta^n\end{array}\right\} / (\text{degen obj}) \qquad T(\Delta^k)$$

$$\uparrow \qquad\qquad\qquad\qquad \updownarrow$$

$$(\mathcal{C}^n)_k \qquad\qquad\qquad\qquad \Delta^k$$

$$T = \text{tot}\left(\partial_k\Delta^k \to \partial_{k-1}\Delta^k \to \cdots \to \partial_0\Delta^k\right)$$

Features:

$$N(B[n]) \simeq K(B,n)$$ "already" EM spec.

- $K(B,1) = S_\bullet(B)$ S-dot construction of B!

- $K(B,2) = S_\bullet^{2,1}(B)$ Hesselholt-Madsen "real $S_\bullet$ construction" used for $\mathbb{R}$ K-thy.

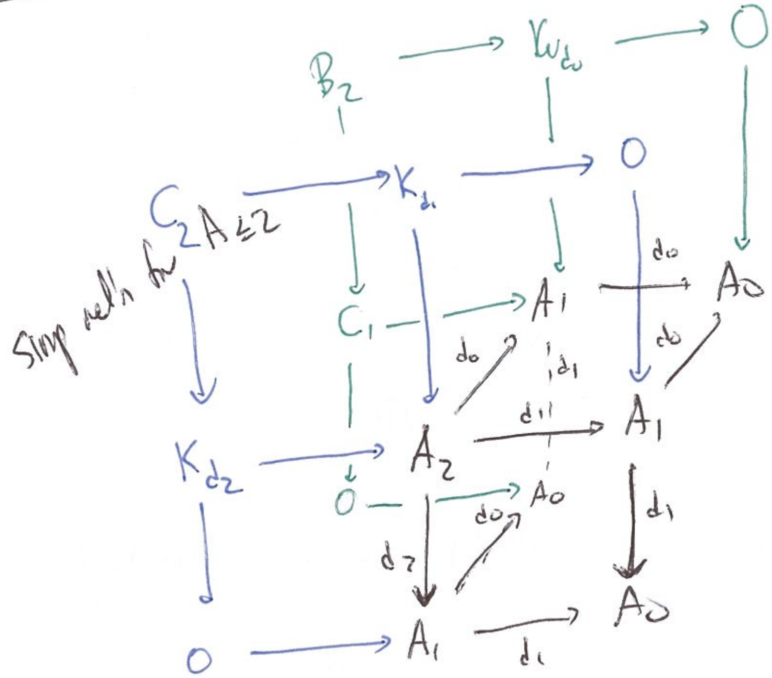
- $K(B,n)$ could be called "$S_\bullet^{n,1}(B)$", considered by G. Jasso for n-abelian cats

- $N(B_1 \xrightarrow{f} B_2) = S_\bullet(f)$ S-dot rel f.

- $K(B,n)$ satisfy higher analogue of 2-Segal properties.

  odd n-Segal spaces

Dyckerhoff talk scratchwork



Pattern:

$$\left(\bigcup_{i=1}^{n} A_n \xrightarrow{d_i} A_{n-1}\right) \xrightarrow{d_0} \left(\bigcup_{j=0}^{n-1} A_{n-1} \xrightarrow{d_j'} A_{n-2}\right)$$

$$\mathrm{Fun}\left(\bigvee_n \Delta^1, \mathrm{Cat}_\infty\right) \supset \qquad\qquad \supset \mathrm{Fun}\left(\bigvee_{n-1} \Delta^1, \mathrm{Cat}_\infty\right)$$

$$d_0 d_i = d_{i-1} d_0$$

$$\begin{tikzcd} & A_{n-1} \arrow[d, "d_{i-1}"] \\ A_n \arrow[ur, "d_0"] \arrow[d, "d_i"'] \arrow[r, phantom, "\circlearrowleft"] & A_{n-2} \\ A_{n-1} \arrow[ur, "d_0"'] & \end{tikzcd}$$

This induces map

$$\left(\bigcup_{i=1}^{n} A_n \xrightarrow{d_i} \underset{\displaystyle\overset{\downarrow}{0}}{A_{n-1}} \right) \xrightarrow[\substack{\text{or} \\ 0}]{d_0} \left(\bigcup_{j=0}^{n-1} A_{n-1} \xrightarrow{d_j} \underset{\displaystyle\overset{\downarrow}{0}}{A_{n-2}} \right)$$

$$\begin{matrix} \cap & & \cap \\ \underbrace{\mathrm{Fun}\left(\bigvee_n \Delta^1 \cup_1 \Delta^1, \mathrm{Cat}_\infty \right)}_{\underline{C}_n} & & \underbrace{\mathrm{Fun}\left(\bigvee_n \Delta^1 \cup_1 \Delta^1, \mathrm{Cat}_\infty \right)}_{\underline{B}_n} \end{matrix}$$

hence map

$$\begin{matrix} C_n & \xrightarrow{\text{"}d_0\text{"}} & B_n \\ \parallel & & \parallel \\ \lim \underline{C}_n & & \lim \underline{B}_n . \end{matrix}$$

$$\begin{tikzcd} \lim \underline{B}_n \arrow[d, dashed] \arrow[dr, bend left] & \\ \lim C_{n-1} \arrow[d] & B_n \\ C_n & \end{tikzcd}$$

We also have map $B_n \to C_{n-1}$ because of inclusion

$$\left(\bigcup_{j=1}^{n-1} A_{n-1} \xrightarrow{d_j} \underset{\displaystyle\overset{\downarrow}{0}}{A_{n-2}} \right) \hookrightarrow \left(\bigcup_{j=0}^{n-1} A_{n-1} \xrightarrow{d_j} \underset{\displaystyle\overset{\downarrow}{0}}{A_{n-2}} \right)$$

$$\begin{matrix} \mathrm{Cat}_\infty \\ \uparrow \underline{B}_n \end{matrix}$$

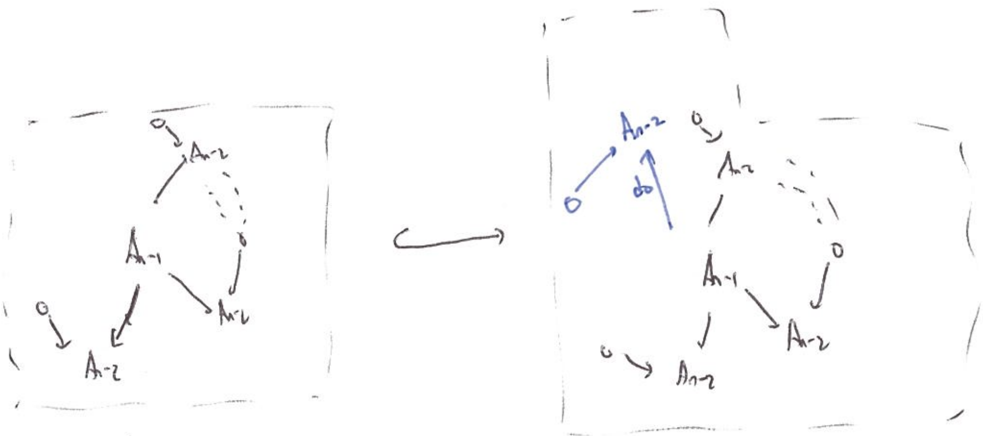
more rigorously, $\underline{C_{n-1}} = \alpha^* \underline{B_n}$ where $\alpha : \left(\bigvee_{n-1} \Delta^1 \cup_1 \Delta^1 \right) \hookrightarrow \left(\bigvee_n \Delta^1 \cup_1 \Delta^1 \right)$

so have $\lim \underline{B_n} \longrightarrow \lim \alpha^* \underline{C_{n-1}}$.

So the map

$$
\begin{tikzcd}
C_n \arrow[r, "\text{"}d_0\text{"}"] \arrow[rr, bend right, "\text{"}d_0\text{"}"'] & B_n \arrow[r, "\text{"}\alpha^*\text{"}"] & C_{n-1}
\end{tikzcd}
$$

is what Toby calls d_0 in the talk. Why does $d_0 \circ d_0 \sim 0$?
Well, consider that



$\underline{C_{n-1}}$ $\underline{B_n}$

e.g.

$$
\begin{tikzcd}
A_1 \arrow[d, "d_1"] & & A_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & A_0 \\
0 \arrow[r] & A_0 \arrow[r, hook] & 0 \arrow[r] & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 0 \arrow[d] \\
 A_0
\end{tikzcd}
$$

so by iterated pb property, $B_n \simeq$ p.b. $\left(\begin{tikzcd} & 0 \arrow[d] \\ C_{n-1} \arrow[r, "d_0"] & A_{n-1} \end{tikzcd} \right)$.

Study composition

$$\begin{tikzcd}
C_n \arrow[r, "\text{``}d_0\text{''}"] & B_n \arrow[r, "\alpha^*"] \arrow[d, "pb"] & C_{n-1} \arrow[r, "\text{``}d_0\text{''}"] \arrow[d, "d_0"] & B_{n-1} \arrow[r, "\alpha^*"] \arrow[d, "pb"] & C_{n-2} \arrow[d, "d_0"] \\
 & 0 \arrow[r] & A_{n-1} & 0 \arrow[r] & A_{n-2}
\end{tikzcd}$$