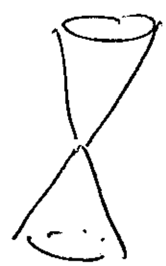
Frégier |

"Every deformation problem is controlled by a dgla" ~ actually kind of wrong as stated.

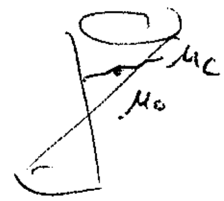
1) Illustration. Gerstenhaber def on assoc alg.

Fix V. $\{\text{assoc alg str}\} = As$ looks quadratic:



$$\begin{array}{c} \| \\ \{V\otimes V \xrightarrow{\mu} V \text{ assoc}\} \\ \cap \\ L(V^{\otimes 2}, V). \end{array}$$

Given μ_t a curve in As, it's like a deformation of μ_0.



Q1): How to describe moduli problem algebraically?

2) " defm of μ " ?

A1): $As = \{\mu \in L^1 \text{ st } [\mu,\mu]_G = 0\}$ — $L^\bullet$ graded Lie alg, $[\,,\,]_G$ Gerstenhaber bracket

$$L^\bullet = \oplus L^i, \quad L^i = L(V^{\otimes i+1}, V)$$

$$[\,,\,]_G : L^i \otimes L^j \to L^{i+j} \quad (\deg 0)$$

$$[a,b] = (-1)^{ab}[b,a]$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{ab}[b,[a,c]].$$

A2) Twist by Maurer-Cartan element: Want $\mu_0 + \tilde\mu$ to be another assoc alg. Equiv MC eqn.

$$0 = [\mu_0+\tilde\mu, \mu_0+\tilde\mu] = \underbrace{[\mu_0,\mu_0]}_{0} + 2[\mu_0,\tilde\mu] + [\tilde\mu,\tilde\mu] = \overbrace{d_{\mu_0}\tilde\mu + \tfrac{1}{2}[\tilde\mu,\tilde\mu] = 0}^{\text{MC eqn.}}$$

2) DGLA aren't good enough.   cubic   quad.

Given $\mu \xrightarrow{\varphi} \nu$, the eqn should be $\nu(\varphi u_1, \varphi u_2) = \varphi(\mu(u_1, u_2))$

i.e, $(U,\mu) \xrightarrow{\varphi} (V,\nu)$ have $\alpha = \begin{pmatrix} \mu \\ \nu \\ \varphi \end{pmatrix}$, so LHS cubic, RHS quadratic.

But As eqn was quadratic — can use Lie bracket, only two terms necessary.
$\underbrace{\text{Lie bracket}}_{\text{binary bracket}}$

So cubic term suggests we need ternary bracket

$$d\alpha + \frac{1}{2}\{\alpha,\alpha\} + \frac{1}{3}\{\alpha,\alpha,\alpha\} \cdots = 0.$$

ie, need L_∞ alg, to encode deformations of (μ, ν, φ).

3) $L_\infty[1]$ algebras.   V-data   (V for Voronov)

Is a collection
- $L = \oplus L^i$ Graded Lie alg.
- $\mathfrak{a} \subset L$ $[\mathfrak{a},\mathfrak{a}] = 0$, ie, abelian Lie subalg.
- $P : L \to \mathfrak{a}$ projection, $P^2 = P$
- $\Delta \in \ker^1$; deg 1 elt, st $[\Delta,\Delta] = 0$

$\ker^1 = \ker P \cap L^1$

An $L_\infty[1]$ alg is given by ~~$\mathfrak{a}$ $\{a_1,\cdots,a_n\}$~~ $a_1,\cdots,a_n \in \mathfrak{a}$, $\{a_1,\cdots,a_n\} = P([\cdots[\Delta,a_1],\cdots,a_n])$

Thm (Voronov) $\{\cdots\}$ defines an L_∞ alg.   — Call it $\mathfrak{a}^P_\Delta$.

In general,
L = space of cochains
$\mathfrak{a}$ = underlying L_∞ alg

Prop (Voronov)

Given V-data, consider

$$L[1] \oplus \mathfrak{a}. \qquad D = [\Delta, -].$$

This is $L_\infty[1]$-alg w/ brackets

$$(x[1], a) \underset{d = l_0}{\longmapsto} (\pm D x[1],\ p(x + Da))$$

$$\{x[1], y[1]\} = (-1)^{x}[x,y][1]$$

$$\{a_n, \dots, a_1\} = p([\dots [\Delta, a_1], \dots, a_n]).$$

$$\{x, a_1, \dots, a_n\} = p([\dots [x, a_1], \dots, a_n]).$$

Call this $(L[1] + \mathfrak{a})^{P}_{\Delta}$.

4) Apply to $(U,\mu) \xrightarrow{\varphi} (V,\nu)$. V-data given by

$$L = \bigoplus \mathrm{Hom}\left((U \oplus V)^{\otimes i+1}, U \oplus V\right)$$

$$\mathfrak{a} = \bigoplus \mathrm{Hom}\left(U^{\otimes i+1}, V\right)$$

$P : L \to \mathfrak{a}$ projection to component

$\Delta = \mu + \nu$

No dependence on φ yet.

Lemma Given $\varphi : \mu \to \nu$ is a morphism iff

$$\varphi \in MC(\mathfrak{a}^{P}_{\Delta})$$

Rmk $\mathcal{P} = \mathrm{Lie}$, then

$$\underbrace{\mathfrak{g} \xrightarrow{\Phi} \mathfrak{h}}_{\text{deform}} \longmapsto \mathfrak{h}^* \xrightarrow{\varphi^*} \mathfrak{g}^*$$

Lie $\qquad$ Poiss

$\Updownarrow$

$\Gamma(\varphi^*)$ is coisotropic in $\mathfrak{h}^* \times \mathfrak{g}^*$.

So gives deformations of coisotropics in $\mathfrak{h}^* \times \mathfrak{g}^*$.

Prop

$$\left.\begin{array}{l} \overbrace{[\Delta+\tilde{\Delta}, D+\tilde{D}]=0}^{\text{defining alg.}} \\ \underbrace{\varphi+\tilde{\varphi}\in MC(\mathfrak{g}^{P}_{\Delta+\tilde{\Delta}})}_{\text{defining alg map}} \end{array}\right| \iff \tilde{\Delta}+\tilde{\varphi}\in MC\left(\left(L[1]+\mathfrak{a}\right)^{P\varphi}_{\Delta}\right)$$

$$P_\varphi := P\circ e^{[-,\varphi]} = P\circ \mathrm{ad}\varphi.$$

5] Application to other algebras.

In P-algebra, where P operad. Assume P Koszul operad.

$$L_G \simeq \left(\mathrm{Coder}(\underbrace{TV}_{\text{tensor coalg on } V}), [\,,\,]\right) \qquad \text{for As.}$$

Coderivations. Can commute, so get bracket.

Part: TV = Free assoc alg on V. Importantly, Koszul dual $As^{i} =$ As coalg.

Given P, can construct free P-alg PV. Here, we must take $P^{i}(V)$ free P^{i} coalg on V, P^{i} Koszul dual.

Coalg $\Rightarrow$ can speak of Coder.

$$L_G \simeq \left(\mathrm{Coder}(P^{i}(V)), [\,,\,]\right).$$

In general, for $I \to$ PAlg, have Gerstenhaber-Schack cohomology. Instead of $[\,,\,]$, do Grothendieck construction. Given $D: I \to \mathcal{C}$, $\exists$ assoc alg "$!D$", different kind of "shreek"

Def thy of D is that of $!D$. Since $!D$, has dgla, $[\,,\,]_G$ assoc. to it.

Believed that dgla of $!D_\alpha$ is equiv to L_∞ alg of D constructed before.