Krause: Local duality for representations of finite group schemes

Über die Bedeutung des Kommutativen für das Nichtkommutative
(Noether ICM 1932)

G finite group scheme / k
$\rightsquigarrow$ kG, f.d. cocommutative Hopf algebra
Mod G, mod G $\quad$ ($G = kG$?)
StMod G
$H^*(G,k) = \mathrm{Ext}^*_G(k,k)$ graded commutative, f.g. as k-algebra (Friedlander–Suslin)

Theorem: [BIKP] The $H^*(G,k)$-linear triangulated category StMod G is Gorenstein, therefore it has local duality

unclear

i.e. local cohomology objects $\cong$ injective cohomology objects
up to twist and shift

Example A commutative noetherian ring (local property)
A Gorenstein $\Leftrightarrow$ $\mathrm{injdim}_{A_\mathfrak{p}} A_\mathfrak{p} < +\infty \quad \forall \mathfrak{p} \in \mathrm{Spec}\, A$
$D(A) \rightleftarrows D(A_\mathfrak{p})$ is a localisation
$\mathfrak{p}$-local objects

$D(A) \overset{(-)_\mathfrak{p}}{\rightleftarrows} D(A_\mathfrak{p}) \underset{\Gamma_\mathfrak{p}}{\rightleftarrows} D_{\mathfrak{p}\text{-tors}}(A_\mathfrak{p})$
$X \mapsto X_\mathfrak{p} \mapsto \Gamma_\mathfrak{p} X$ = local cohomology of X

A Gorenstein $\overset{!}{\Leftrightarrow}$ $\Gamma_\mathfrak{p}(A) \cong \Sigma^{-\dim A_\mathfrak{p}} E(A/\mathfrak{p})$
injective envelope of $A/\mathfrak{p}$
(= concentrated in 1 degree)

What does this mean for StMod G?

StMod G is a triangulated category, $\Sigma = \Omega^{-1}$ (inverse syzygy)
and $H^*(G,k)$-linear, i.e.

$$\underline{\mathrm{Hom}}^*_G(X,Y) = \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_G(X, \Omega^n Y)$$

is a module over $H^*(G,k)$. It is given by

$-\otimes_k X : H^*(G,k) \to \underline{\mathrm{End}}^*_G(X,X)$

Denote $(\mathrm{StMod}\,G)_{\mathfrak{p}} = \{ X \in \mathrm{StMod}\,G \mid \underline{\mathrm{Hom}}^*(C,X)$ $\mathfrak{p}$-local $\forall C \in \mathrm{mod}\,G \}$
for $\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$

Denote $\Gamma_{\mathfrak{p}}\,\mathrm{StMod}\,G = \{ X \in (\mathrm{StMod}\,kG)_{\mathfrak{p}} \mid \underline{\mathrm{Hom}}^*(C,X)$ $\mathfrak{p}$-local $\mathfrak{p}$-torsion $\forall C \in \mathrm{mod}\,G \}$

<u>Remark:</u> $\Gamma_{\mathfrak{p}}\,\mathrm{StMod}\,G$ are the minimal localising subcategories of StMod G

$$\begin{array}{ccccc} \mathrm{StMod}\,G & \rightleftarrows & (\mathrm{StMod}\,G)_{\mathfrak{p}} & \to & \Gamma_{\mathfrak{p}}\,\mathrm{StMod} \\ X & \mapsto & X_{\mathfrak{p}} & \mapsto & \Gamma_{\mathfrak{p}} X \end{array}$$

The local cohomology object $\Gamma_{\mathfrak{p}}(k)$ or $\simeq{}^{t}\kappa_{\mathfrak{p}}$ is the <u>Rickard idempotent</u> ($\to$ wrt tensor product)

For $C \in \mathrm{mod}\,G$, $\mathrm{Hom}^*_R\left(\underline{\mathrm{Hom}}^*_G(C,-), E(R/\mathfrak{p})\right)$ $\quad R = H^*(G,k)$

$\overset{\text{Brown rep}}{\cong} \underline{\mathrm{Hom}}_G(-, T_{\mathfrak{p}} C)$

The injective cohomology object is $T_{\mathfrak{p}}(k)$

The <u>twist</u> comes from the Nakayama functor:

$$\nu(X) = X \otimes_{kG} kG^{\vee} = X \otimes_k \underbrace{\delta_G}_{\text{modular character}}$$

(for G a group: kG symmetric) $\quad$ (for G a group: $\delta_G = k$)

Theorem: [BIKP] $\boxed{\Gamma_{\mathfrak{p}} \circ \mathbb{D} \cong \Omega^{-d} \circ T_{\mathfrak{p}}}$

$d = \dim \operatorname{Proj} R/\mathfrak{p}$

Local duality

Serre duality for smooth proj. X/k, $\dim X = n$, says $\forall\, \mathcal{F} \in \operatorname{coh} X$:

$$\boxed{\operatorname{Ext}^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^{\vee}}$$

$$= \operatorname{Hom}_k(H^{n-i}(\mathcal{F}), k)$$

dualizing object $\qquad$ Matlis duality (over k)

Now:

Theorem $\forall$ G-module M, $\mathfrak{p} \in \operatorname{Proj} R$, $i \in \mathbb{Z}$

Tate-Ext

$$\boxed{\widehat{\operatorname{Ext}}^i_G(M, \Gamma_{\mathfrak{p}}(\mathcal{J}_G)) \cong \operatorname{Hom}_R\left(H^{*-d-i}(G, M), I(\mathfrak{p})\right)}$$

for $I(\mathfrak{p}) = E(R/\mathfrak{p})$

Serre functors

$D^b(\operatorname{coh} X)$ as k-linear triangulated category has one

Problem $D^b(\operatorname{mod} G)$, $\operatorname{stmod} G$ has no Serre functor as k-linear category (only if kG is semisimple)

Theorem (BIKP)

$D^b(\operatorname{mod} G) \xrightarrow[\cong]{\mathbb{D}} D^b(\operatorname{mod} G)$ induces local Serre duality, i.e.

$T = D^b(\operatorname{mod} G)$, $R = \operatorname{Ext}^*(G, k)$

$$\begin{array}{ccccc} T & \longrightarrow & T_{\mathfrak{p}} & \longleftarrow & \Gamma_{\mathfrak{p}} T_{\mathfrak{p}} \\ {\scriptstyle \mathbb{D} = F}\downarrow\cong & & \cong\downarrow F_{\mathfrak{p}} & & \downarrow F_{\mathfrak{p}} \\ T & \longrightarrow & T_{\mathfrak{p}} & \longleftarrow & \Gamma_{\mathfrak{p}} T_{\mathfrak{p}} \end{array}$$

$\mathfrak{p}$-torsion, i.e. $\operatorname{End}^*_{\Gamma_{\mathfrak{p}} T}(X)$ $\mathfrak{p}$-torsion

R-linear

$F_{\mathfrak{p}} = - \otimes_k \Gamma_{\mathfrak{p}}(\mathcal{J}_G)$

Then $\boxed{\operatorname{Hom}_R\left(\operatorname{Hom}^*_{\Gamma_{\mathfrak{p}} T}(X, Y), I(\mathfrak{p})\right) \cong \operatorname{Hom}_{\Gamma_{\mathfrak{p}} T}\left(Y, \Sigma^{-d} F_{\mathfrak{p}}(X)\right)}$

= Serre functor! $\Leftrightarrow$ Auslander-Reiten triangles