Kuznetsov

Categorical joins + HPD

HPD $\underset{\text{smth}}{X} \xrightarrow{f} \mathbb{P}(V)$, $\underset{\text{smth}}{Y} \xrightarrow{g} \mathbb{P}(V^*)$

In genl, at least one of X, Y is smth.

Fix Lefschetz decomp: symm wrt $\mathcal{O}(1)_{\mathbb{P}(V)}$ pulled back

a semi.⊥ decomp "compatible" w/ f.

$$D(X) \simeq \langle A_0, A_1(1), \dots, A_{m-1}(m-1) \rangle$$

$A_0 \supset A_1 \supset \dots \supset A_{m-1}$ chain of admissible subcats

$$\underset{A_1(1)}{\text{twist by } f^*\mathcal{O}(1)} \Big\} \underset{A_{m-1}(m-1)}{\text{twist } m-1 \text{ times}}$$

and

$$D(Y) \simeq \langle B_{n-1}(1-n), \dots, B_0 \rangle$$

$$B_{n-1} \subset B \cdots \subset B_1 \subset B_0.$$

Rmk Lef. decomp always det. by A_0, or by B_0. eg: $A_1(1) \simeq (A_0^{\perp})$ inside $A_0(1)$.

Main HPD thm:

Thm ∃ relation btw D(linear sections of X) & of Y.

$$D(X_L) = \langle \mathcal{C}_L, A_r(r), \dots, A_{m-1}(m-1) \rangle$$

$$D(Y_L) = \langle B_{n-1}(1-n), \dots, B_{N-r}(N-r), \mathcal{C}_L \rangle$$

where $N = \dim V$.

part: $\mathcal{C}_L$ are same. So getting $D(X_L)$ tells you abt $D(Y_L)$

ex Y_L empty $\Rightarrow \mathcal{C}_L = 0$, so get $\mathcal{D}(X_L) = \langle \mathcal{A}_r(r), \ldots, \mathcal{A}_{m-1}(m-1) \rangle$.

HPD is most powerful tool we have to describe $\mathcal{D}$, eg, for Fanos.

Ex
- $X = \mathbb{P}_S(E)$ S smth proj,

  assume $E \hookrightarrow V \otimes \mathcal{O}_S$ embeds in trivl bdl $V \otimes -$.

  Get $\mathbb{P}_S(E) \to \mathbb{P}(V)$. Previous work shows $\mathcal{D}(X)$ is of form we want.

  The main problem of HPD is to construct Y — such Y is unique.

  It's hard, but gives strong consequences.

  Here, turns out $Y = \mathbb{P}_S(E^\perp)$, $E^\perp := \ker(V^\vee \otimes \mathcal{O}_S \to E^\vee)$.

$$\begin{array}{c} Y = \mathbb{P}_S(E^\perp) \\ \downarrow \\ \mathbb{P}(V^\vee) \end{array}$$

  This HPD is called "linear duality"

- Quadrics. Assume $X \subset \mathbb{P}(V)$, $X = Q$ quadric, even-dimensional.

  Lefschetz decomp: $\mathcal{A}_0 = \langle \mathcal{O}, S_Q \rangle$

  a Spinor bdl (one of two) on Q.

  Then $Y = Q^\vee \subset \mathbb{P}(V^\vee)$.

  "Quadratic duality."

  When $\dim Q$ odd, answer differs: $Y \xrightarrow{2:1} \mathbb{P}(V^\vee)$ double cover branched over $Q^\vee$.

  Y smth but higher-dim! Related to classical duality but in different way

- $X = \mathbb{P}(W) \xhookrightarrow{v_2} \mathbb{P}(S^2 W)$ then $Y = (\mathbb{P}(S^2 W^\vee), \mathcal{Cl}_0)$

  double Veronese emb — shef of Clifford algebras. — Marcomm example.

- $X = Gr(2, W) \hookrightarrow \mathbb{P}(\Lambda^2 W)$ then $\cancel{\ } \; Pf(W) \subset \mathbb{P}(\Lambda^2 W^\vee)$ ~ singular

  Plucker embedd

  $Y = \widetilde{Pf}(W)$ ~ some smoothing. Pf = Pfaffian.

[So want to find: Given HPD pair (X, Y) produce a new pair!]

Example operation:

Assume g not surjective, so $\varphi \in \mathbb{P}(V^\vee) \setminus g(Y)$. Set $V' := V_\varphi$ = annihilator of φ (φ a linear fn) $= \ker \varphi \subset V$.

Then $X' := X \times_{\mathbb{P}(V)} V' \xrightarrow{f'} \mathbb{P}(V')$.

and we find $Y' = Y \to \mathbb{P}(V^\vee)$

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "g'"'] & \mathbb{P}(V^\vee) \arrow[d, dashed, "pr_\varphi"] \\ & \mathbb{P}((V')^\vee) \end{tikzcd}$$

linear proj centred @ φ; birational; g' is regular map since pr_φ is honest map restricted to Y.

More gnlly, $Y' = Bl(Y)$ along some "centre" of fiber at φ.

Nothing really new from this — X', Y' have same linear sections as X, Y.

Moreover, (X', Y') "simpler" than (X, Y).

The "categorical join" operation is one cojectural operation.

Joins

Assume $X_i \overset{f_i}{\underset{\text{emb.}}{\hookrightarrow}} \mathbb{P}(V_i)$ (embeddings, for now)

Then $$J(X_1, X_2) = \text{join} := \bigcup_{\substack{\text{lines} \\ \text{connecting} \\ x_i \in X_i}} \langle x_1, x_2 \rangle \subset \mathbb{P}(V_1 \oplus V_2)$$

Very classical operation. Unpleasant: J is usually singular. (Take $X_2 = pt$; get cone $(X_1) = CX_1$.)

~~In general,~~ (usually) it's singular locus is $X_1 \cup X_2$.

exception: If X_1, X_2 are linear subvarieties, then $J(X_1, X_2)$ is smth.

or $X_i = \mathbb{P}(V_i)$, then $J(X_1, X_2) = \mathbb{P}(V_1 \oplus V_2)$

So J not in HPD setup.
Need resol'n of sing. Have a canonical one! Replace line by abstract lines. Move lines apart from each other.

Defn Resolved join $\tilde{J}(X_1, X_2) := \mathbb{P}_{X_1 \times X_2}(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$

exterior tensor product $\mathcal{O}_{X_1}(-1) \boxtimes \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_1} \boxtimes \mathcal{O}_{X_1}(-1)$

Have $$\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \hookrightarrow (V_1 \oplus V_2) \otimes \mathcal{O}$$

so have emb to $\mathbb{P}(V_1 \oplus V_2)$. $\tilde{J}$ smth b/c proj bundle over $X_1 \times X_2$, smth.

Still unpleasant propty

$$\begin{tikzcd} \tilde{f}: \tilde{J}(X_1,X_2) \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{P}(V_1 \oplus V_2) \\ J(X_1,X_2) \arrow[ur] & \end{tikzcd}$$

$\cong$ away from single locus

have two exceptional divisors — one over X_1, one over X_2

each is $\cong$ to $X_1 \times X_2$

first: $\mathbb{P}_{X_1\times X_2}(\mathcal{O}(-1,0)) \cong X_1 \times X_2 =: E_1$

$$\begin{tikzcd} X_1 & \mathbb{P}_{X_1\times X_2}(\mathcal{O}(-1,0)) \arrow[l, "\tilde{f}"'] \arrow[d, hook, "\varepsilon_1"] \\ & \tilde{J} \\ X_2 & \mathbb{P}_{X_1\times X_2}(\mathcal{O}(0,-1)) \arrow[l, "\tilde{f}"] \arrow[u, hook, "\varepsilon_2"'] \end{tikzcd}$$

$\mathbb{P}_{X_1\times X_2}(\mathcal{O}(0,-1)) \cong X_1 \times X_2 =: E_2$

$\tilde{J}$ is big, "too big." eg: • J already smooth $\Rightarrow$ $\tilde{J}$ much bigger $Bl_{\cup X_1\times X_2}(\mathbb{P})$.

• HPD requires some mixture, w/ contributions from excep divisors

A smaller, non-comm resolution exists if one is given Lef. Decomp for X_i:

Assume • f_i arbitrary (not nec ~~smth~~ emb)

• $D(X_i) = \langle A_{i0}, \dots \rangle$

Categorical res. of J sits inside $D(\tilde{J})$.

Defn $\mathcal{J}(X_1,X_2) = \{ F \in D(\tilde{J}(X_1,X_2))$ s.t. $F|_{E_1} = \varepsilon_1^* F \subset D(X_1) \boxtimes A_{20}$, $\varepsilon_2^* F \subset A_{10} \boxtimes D(X_2) \}$

categorical join

in derived sense

(Obv, depends on choice of A_{i0})

This is admissible subcat, linear over $\mathbb{P}(V_1 \oplus V_2)$, leads to semiorthog. decomp of $D(\tilde{\mathcal{J}})$. So can view as non-comm var over $\mathbb{P}(V_1 \oplus V_2)$:

Prop 1) $\mathcal{J}$ is • admissible in $D(\tilde{\mathcal{J}})$

• $\mathbb{P}(V_1 \oplus V_2)$-linear.

(ie, $\mathcal{J}$ is smth + proper non-comm var over $\mathbb{P}(V_1 \oplus V_2)$.)

Ex $X_i = \mathbb{P}(V_i)$

Then $\tilde{\mathcal{J}} = \mathrm{Bl}(\mathcal{J})$

And $\mathcal{J}(X_1, X_2) = D(\mathbb{P}(V_1 \oplus V_2))$. (If A_{i0} = Beilinson collections, can do this.)

2) $\exists$ natural Lefschetz decomp, via

$$A_0 := \pi^*(A_{10} \boxtimes A_{20})$$

$$\tilde{\mathcal{J}} \xrightarrow{\pi} X_1 \times X_2$$

(π $\mathbb{P}^1$ fibration $\Rightarrow \pi^*$ fully faithful.)

(note $\varepsilon_i^* \pi^* = \mathrm{id}$, so easy.)

(Note $\boxtimes$ of L.d. of X_i gives semiorth decomp of $X_1 \times X_2$.)

What is HPD of $\mathcal{J}$?

Conj $\mathcal{J}(Y_1, Y_2) \longrightarrow \mathbb{P}(V_1^\vee \oplus V_2^\vee)$ is HPD to $\mathcal{J}(X_1, X_2)$.

Rmk X_i can be non-comm; obvious analogues of this geom constr exist.

Ex If $\varphi: V_1 \cong V_2$, $\mathbb{P}(\Gamma_\varphi) \subset \mathbb{P}(V_1 \oplus V_2)$. Can check $\mathcal{J}(X_1, X_2) \times_{\mathbb{P}(V_1 \oplus V_2)} \mathbb{P}(\Gamma_\varphi) \cong X_1 \times_{\mathbb{P}(V_2)} X_2$

$$\mathcal{J}(X_1,X_2) \times_{\mathbb{P}(V_1 \oplus V_2)} \mathbb{P}(\Gamma_\varphi) \cong$$

$\Gamma_\varphi^\perp = \Gamma_{\varphi^\vee}$, can do this out. HPD for X_1, X_2 gives decomp of intersection's D^b, too!