

Kuznetsov

Categorical joins + HPD

HPD

$$X \xrightarrow{f} \mathbb{P}(V), \quad Y \xrightarrow{g} \mathbb{P}(V^{\otimes 2})$$

smth                      smth

In good, at least one of  $X, Y$  is smth.

Fix Lelek's decomp: symm wrt  $\mathcal{O}(1)_{\mathbb{P}(V)}$  pulled back  
a semi- $\perp$  decomp "compatible" w/  $f$

$$D(X) \triangleq \langle A_0, A_1(1), \dots, A_{m-1}(m-1) \rangle$$

$A_0 \supset A_1 \supset \dots \supset A_{m-1}$  chain of admissible subcs

$\left. \begin{array}{l} \text{start by } \mathbb{P}(\mathcal{O}(1)) \\ A_1(1) \end{array} \right\} \text{start } m-1 \text{ times}$   
 $A_{m-1}(m-1)$

and

$$D(Y) \triangleq \langle B_{n-1}(1-n), \dots, B_0 \rangle$$

$$B_{n-1} \subset B_{n-2} \subset \dots \subset B_1 \subset B_0$$

Rank Left decomp always def. by  $A_0$ , or by  $B_0$ . eg:  $A_1(1) \triangleq (A_0^{\perp})$  inside  $A_0(1)$ .

Main HPD thm:

Thm  $\exists$  relation btw  $D$  (linear sections of  $X$ ) or of  $Y$ .

$$D(X) = \langle \mathcal{O}_L, A_1(r), \dots, A_{m-1}(m-1) \rangle$$

$$D(Y) = \langle B_{n-1}(1-n), \dots, B_{N-r}(N-r), \mathcal{O}_L \rangle$$

where  $N = \dim V$ .

part:  $\mathcal{O}_L$   
are same  
So getting  
 $D(X)$  tells you  
-...  $D(Y)$

ex  $Y_L$  empty  $\Rightarrow C_L = 0$ , so get  $D(X_L) = (A_1(r_1), \dots, A_{m-1}(r_{m-1}))$

HPD is most powerful tool we have to describe  $D$ , eg, for Fano's

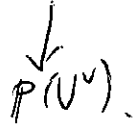
Ex •  $X = \mathbb{P}_S(E)$   $S$  smth proj,

assume  $E \hookrightarrow V \otimes \mathcal{O}_S$  embeds in triv bdl  $V \otimes -$ .

Get  $\mathbb{P}_S(E) \rightarrow \mathbb{P}(V)$ . Previous work shows  $D(X)$  is of form we want

The main problem of HPD is to construct  $Y$  — such  $Y$  is unique. It's hard, but gives strong consequences.

Here, turns out  $Y = \mathbb{P}_S(E^\perp)$ ,  $E^\perp := \ker(V^\vee \otimes \mathcal{O}_S \rightarrow E^\vee)$



This HPD is called "linear duality"

• Quadrics. Assume  $X \subset \mathbb{P}(V)$ ,  $X = Q$  quadric, even-dimensional.

Leiberman desc:  $A_S = \langle \mathcal{O}_S, S \otimes \mathcal{O}_S \rangle$

a Spinor bdl (one of two) on  $Q$ .

Then  $X = Q^\vee \subset \mathbb{P}(V^\vee)$ .

"Quadratic duality."

When  $\dim Q$  odd, answer different:  $Y \xrightarrow{2:1} \mathbb{P}(V^\vee)$  double cover branched

$Y$  smth but higher-dim! Related to charact of  $Q^\vee$  over  $\mathbb{Q}$  but in different way

•  $X = \mathbb{P}(W) \xrightarrow{v_2} \mathbb{P}(S^2 W)$  then  $Y = (\mathbb{P}(S^2 W^v), C_0)$   
 double Veronese emb

shut of Clifford algebra.  
 Marzani example.

•  $X = Gr(2, W) \hookrightarrow \mathbb{P}(1^2 W)$  then  
 Plucker embed

~~$\mathbb{P}F(W) \subset \mathbb{P}(1^2 W^v)$~~   
 $\sim$  singular

$Y = \widetilde{\mathbb{P}F(W)}$

$Pf = Pf_{lin}$

$\sim$  some something

[So want to find: Given HPD pair  $(X, Y)$  produce a new pair!]

Example operation:

Assume  $g$  not surjective, so  $\mathcal{Q} \in \mathbb{P}(V^v) \setminus g(Y)$ . Set  $V' := V_{\mathcal{Q}} = \text{annihilator of } \mathcal{Q}$   
 ( $\mathcal{Q}$  a linear form)  
 $= \ker \mathcal{Q} \subset V$ .

Then  $X' := X \times_{\mathbb{P}(V)} V' \xrightarrow{f'} \mathbb{P}(V')$

and we find  $Y' := Y \rightarrow \mathbb{P}(V^v)$   
 $g' \downarrow \downarrow \text{pr}_{\mathcal{Q}}$   
 $\mathbb{P}(V', V)$

linear proj centered at  $\mathcal{Q}$ ; birational;  $g'$  is reg-m map since  $\text{pr}_{\mathcal{Q}}$  is honest map restricted to  $Y$ .

More quickly,  $Y' = Bl(Y)$  along some "center" of fibers at  $\mathcal{Q}$ .

Nothing really new from this —  $X', Y'$  have same linear sections as  $X, Y$ .

Moreover,  $(X', Y')$  "simpler" than  $(X, Y)$ .

(4)

The "categorical join" operation is one categorical operation.

Joins

Assume  $X_i \xrightarrow[\text{emb.}]{f_i} P(V_i)$  (embeddings, for now)

Then  $J(X_1, X_2) = \text{join} := \bigcup_{\substack{\text{lines} \\ \text{connecting} \\ x_i \in X_i}} \langle x_1, x_2 \rangle \subset P(V_1 \oplus V_2)$

Very classical operation. Unpleasant:  $J$  is usually singular. (Take  $X_2 = \text{pt}$ ; get  $\text{cone}(X_1) = CX_1$ .)

~~It's~~ ~~good~~, it's singular locus is  $X_1 \cup X_2$ .  
(usually)

exception: If  $X_1, X_2$  are linear subspaces, then  $J(X_1, X_2)$  is smooth.

or  $X_i = P(V_i)$ , then  $J(X_1, X_2) \subset P(V_1 \oplus V_2)$

So  $J$  not in HPD setup.

Need results of sing. Have a canonical one! Replace line by abstract lines. Move lines apart from each other.

Def Resolved join  $J(X_1, X_2) := P_{X_1 \times X_2}(\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1))$

(  
exterior tensor product  $\mathcal{O}_{X_1}(-1) \otimes \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_1} \otimes \mathcal{O}_{X_1}(-1)$ )

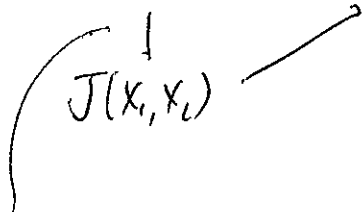
Have  $\mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1) \hookrightarrow (V_1 \oplus V_2) \otimes \mathcal{O}$

so have emb to  $P(V_1 \oplus V_2)$ .  $\exists$  smooth b/c proj bundle over  $X_1 \times X_2$ , smooth.

Still unpleasant property

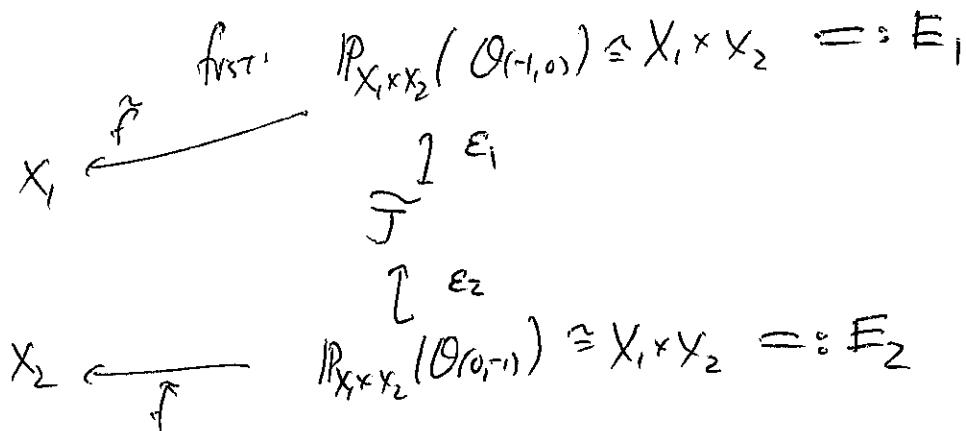
⑤

$$\tilde{f}: \tilde{J}(X_1, X_2) \xrightarrow{\tilde{f}} \mathbb{P}(V_1 \oplus V_2)$$



away from single lines

have two exceptional divisors — one on  $X_1$ , one on  $X_2$   
each is  $\cong$  to  $X_1 \times X_2$



$\tilde{J}$  is big, "too big." eg:  $\tilde{J}$  abnd smth  $\Rightarrow \tilde{J}$  much bigger  $B|_{\cup X_1 \times X_2}(\mathbb{P})$ .

• HPD require some mixture, w/ contributions from excep divisors

A smaller, non-comm resolution exists if one is given Left Decp for  $X_i$ :

Assume  $\bullet f_i$  arbitrary (not nec ~~smth~~ emb)

$$\bullet \mathcal{D}(X_i) = \langle A_{i0}, \dots \rangle$$

Categorical res of  $\tilde{J}$  sits inside  $\mathcal{D}(\tilde{J})$ .

$$\underline{\text{Def}} \quad \mathcal{J}(X_1, X_2) = \{ F \in \mathcal{D}(\tilde{J}(X_1, X_2)) \text{ s.t.} \}$$

$\uparrow$   
categorical join

$$\begin{aligned} \mathcal{F}|_{E_1} &= e_1^* F \subset \mathcal{D}(X_1) \boxtimes A_{20} \\ e_2^* F &\subset A_{10} \boxtimes \mathcal{D}(X_2) \end{aligned}$$

(Obs, depends on choice of  $A_{i0}$ )

This is admissible subcat, linear over  $\mathbb{P}(V_1 \oplus V_2)$ , mod  $\tau$  semiorthog.  
decomp of  $D(\tilde{J})$ . So can view as non-comm var over  $\mathbb{P}(V_1 \oplus V_2)$ :

Prop 7  $f$  is admissible in  $D(\tilde{J})$   
•  $\mathbb{P}(V_1 \oplus V_2)$ -linear.

~~ie~~ (ie,  $f$  is smth non-comm var over  $\mathbb{P}(V_1 \oplus V_2)$ .)  
prop

Ex  $X_i = \mathbb{P}(V_i)$

Then  $\tilde{J} = B(\tilde{J})$

And  $f(X_1, X_2) = D(\mathbb{P}(V_1 \oplus V_2))$ . (If  $A_{i0}$  = Belinson collection, can do this.)

2)  $\exists$  natural Lefschetz decomp, via

$$A_0 := \pi^*(A_{10} \boxtimes A_{20})$$

$$\tilde{J} \xrightarrow{\pi} X_1 \times X_2$$

( $\pi$   $\mathbb{P}^1$ -fibration  $\Rightarrow \pi^*$  fully faithful.)

(note  $e_i^* \pi^* = id$ , so easy.)

(Note  $\boxtimes$  of L.d. of  $X_i$  gives semiorth decomp of  $X_1 \times X_2$ .)

What is HPD of  $f$ ?

Conj  $f(X_1, X_2) \rightarrow \mathbb{P}(V_1 \oplus V_2)$  is HPD to  $f(X_1, X_2)$ .

Rmk  $X_i$  can be non-comm; obvious analogues of this geom also exist.

$$\begin{matrix} \mathcal{J}(X_1, X_2) \times \mathbb{P}(\Gamma_\varphi) \\ \cong \mathbb{P}(V_1 \oplus V_2) \end{matrix}$$

Ex If  $\varphi: V_1 \cong V_2$ ,  $\mathbb{P}(\Gamma_\varphi) \subset \mathbb{P}(V_1 \oplus V_2)$ . Can check  $\mathcal{J}(X_1, X_2) \times \mathbb{P}(\Gamma_\varphi) \cong X_1 \times X_2$   
 $\mathbb{P}(V_1 \oplus V_2)$   $\mathbb{P}(V_2)$

$\Gamma_\varphi \xrightarrow{\downarrow} \Gamma_{\varphi^*}$ , can do this out. HPD for  $X_1, X_2$  gives decomp of intersections  $\mathbb{P}^6$ , too!