Muro: Spaces of locally finite Δ'd cats

Rmk Need k field so H^* is projective, and to use minimal models. Otherwise, use defect tower in the end.

$\frac{1}{2} \in k$ gd field

$\mathcal{T}$ Δ'd cat, idem complete, ess. small, Σ.

really, triangulable; no strctr specified.

Def An algebraic Δ'd str on $\mathcal{T}$ is a dg-cat $\mathcal{C}$, w/ equiv.

$$\mathcal{T} \simeq D^c(\mathcal{C})$$

derv. cat. of cmpct obj.

Then the space of alg. enhancements is

$$\mathcal{M}(\mathcal{T}) := \operatorname{Nerve}\left(\begin{array}{l} \text{ob dg-cat } \mathcal{C} \text{ / } \mathcal{C} \text{ happens to be } D^c(\mathcal{C}) \hat{=} \mathcal{T}. \\ \text{hom Morita fucns} \\ \quad \mathcal{C} \to \mathcal{D} \text{ st} \\ \quad D^c(\mathcal{C}) \simeq D^c(\mathcal{D}) \end{array} \right)$$

Most results are about
- large $\mathcal{T}$, or
- $\mathcal{T} = D(\text{Abelian cat})$.

Note (Toën):

$$\pi_m(\mathcal{M}(\mathcal{T}), \mathcal{C}) \simeq \begin{cases} HH^{2-m}(\mathcal{C}) & m \geq 2 \\ HH^0(\mathcal{C})^\times & m = 2 \\ \text{derived Pic} & m = 1 \end{cases}$$

Note $M(\mathcal{T})$ NOT stack – behaves poorly wrt ext. of scalars

Rmk "cohom fnctr" is one sending exact Δ to ex seqnces.

It's indep of Δ'd structre on $\mathcal{T}$.

Defn $\mathcal{T}$ **locally finite** if any cohom fnctr

$$\mathcal{T} \to Ab \quad \text{or} \quad \mathcal{T}^{op} \to Ab$$

is $\bigoplus$ of ~~dec~~ represntbls

$\Rightarrow$ any indecomp. maps to only finitely many indecomp.

Ex

1) $D^b(\Lambda)$ Λ alg of finite rep type
2) $\underline{\text{mod}}(\Lambda)$ stable module cat Λ self-inj + above hyp.
3) $D^b(\Lambda)/G$ orbit cat, Λ = path alg of Dynkin quiver
4) Cluster cat. of finite type

NOT $D^b(Ab(Cat))$

Δ'd Cat $\longleftrightarrow$ self-inj. algebras

loc finite $\longleftrightarrow$ artinian (noetherian)

Graded enrichment of $\mathcal{T}$: $\mathcal{T}^n(X,Y) := \hom_{\mathcal{T}}(X, \Sigma^n Y)$ ③

Hoch coh of $\mathcal{T}$ bigraded:

$$HH^{s,t}(\mathcal{T})$$

s Hochschild degree

t internal degree (since $\mathcal{T}$ graded)

$$\widehat{HH}^{s,t}(\mathcal{T})$$

Hochschild-Tate cohom, defined b/c
~~only~~ • all projectives are injective
• $\mathcal{T}$ has 2-sided res. over itself
when artinian / noetherian

<u>Claim</u> $\mathcal{T}$ locally finite $\Rightarrow$

$$\pi_0 \mathcal{M}(\mathcal{T}) \cong \Big(HH^{3,-1}(\mathcal{T}) \cap \underbrace{\widehat{HH}^{*,*}(\mathcal{T})^{\times}}_{\text{units}} \cap V([x,x]) \Big)$$

vanishing locus of Gerstenhaber square; the Lie alg str.

modulo $\mathrm{Aut}(\mathcal{T})$.

If $\frac{1}{2} \notin k$, just use Dyer-Lashof, not Gerstenhaber.

Given $[x] \in \pi_0$,

have SES $\quad \circ \hookrightarrow \pi_?(\mathcal{M}(\mathcal{T}), x) \twoheadrightarrow \circ$

where $\circ$ computed by kernel of $[x, -]$ on $HH^{?,?}(\mathcal{T})$

$A_m(\mathcal{T})$ has ob A_m-cat w/ equiv $H^*\mathcal{C} \xrightarrow{\sim} \mathcal{T}$

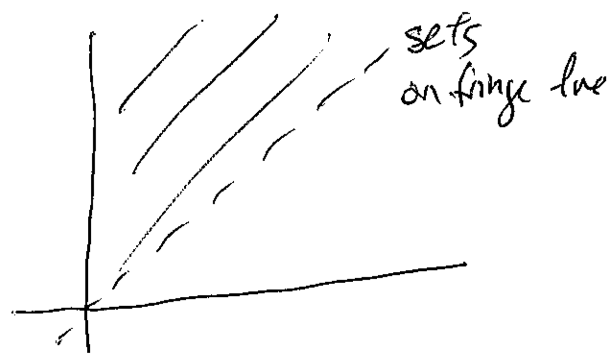
$1 \le m \le \infty$,

have

$$\mathcal{M}'(\mathcal{T}) \xhookrightarrow[\text{full}]{} A_\infty(\mathcal{T}) = \underbrace{\operatorname{holim}_m A_m(\mathcal{T})}_{\text{tower of spaces}}$$

→ Bousfield Kan fringed spec seq $\quad \cdots \to A_{m+1} \xrightarrow[\text{fgt}]{} A_m \to \cdots$

$$E_2^{s,t} \Rightarrow \pi_{t-s}(\mathcal{M}'(\mathcal{T}), \mathcal{C})$$



($\mathcal{M}'(\mathcal{T}) \leftrightarrow \mathcal{M}(\mathcal{T})$, when $\mathcal{M}'(\mathcal{T})$ = ob pre-Δ'd $\mathcal{C}$, hom quis, Toën captures this)
weird category — identify $H^*(\mathcal{E})$ w/ $\mathcal{T}$, but homs only compatible w/ objs, not nec. w/ equivalences. Weird!)

Any enhancement $\mathcal{C}$ of $\mathcal{T}$ has minimal A_∞ model $(\mathcal{T}, m_3, m_4, \ldots)$

where
- $m_3 \in HH^{3,-1}$
- E_2 diff is $\cancel{[x, x]}$ $[\{m_3\}, -]$.

(m_3 univ Massey product; determines all Massey prods of $\mathcal{T}$)