COMPACT MODULI OF MARKED NONCOMMUTATIVE DEL PEZZO SURFACES (1)

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§1. Construction of moduli (via nc $\mathbb{P}^2$) $/k=\bar{k}$, $\mathrm{char}(k)=0$

Thm (Bondal '89, Rickard '89)

X: sm. proj. var. $/k$ $\quad D^b\mathrm{coh}X \ni \mathcal{E}_1,\dots,\mathcal{E}_r$: full strong exc. collection (FSEC) full strong exc. coll.

$\Rightarrow D(X) \simeq D^b\mathrm{mod}\,A$, $\quad A=\mathrm{End}_X(T)$, $T=\bigoplus_{i=1}^{r}\mathcal{E}_i$

A is described via quivers:

e.g. $X=\mathbb{P}^2_{xyz}$, $\quad \mathcal{O},\mathcal{O}(1),\mathcal{O}(2)$ FSEC

$$Q = \begin{tikzcd} 0 \arrow[r, shift left=2, "x_1\, y_1\, z_1"] \arrow[r] \arrow[r, shift right=2] & 1 \arrow[r, shift left=2, "x_2\, y_2\, z_2"] \arrow[r] \arrow[r, shift right=2] & 2 \end{tikzcd}$$

$$kQ/I \xrightarrow[\sim]{\varphi} A=\mathrm{End}(\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))$$

$I=\langle y_2x_1-x_2y_1,\ z_2y_1-y_2z_1,\ x_2z_1-z_2x_1\rangle$

$e_i \longmapsto \mathrm{id}_{\mathcal{O}(i)}$

$x_1 \longmapsto [\mathcal{O}\xrightarrow{x}\mathcal{O}(1)]$

$y_2 \longmapsto [\mathcal{O}(1)\xrightarrow{y}\mathcal{O}(2)]$

Remark: φ homomorphism of $k^3=k^{Q_0}$-alg

$(Q=(Q_0,Q_1,s,t))$

$$\begin{tikzcd} e_i \in kQ/I \arrow[rr] & & A \ni \mathrm{id}_{\mathcal{E}_i} \\ & k^{Q_0} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$e_i \mapsfrom e_i$, $\quad e_i \mapsto \mathrm{id}_{\mathcal{E}_i}$, $\quad e_i=(0,\dots,\overset{i}{1},\dots,0)$

Take $g\in \mathrm{Aut}_{k^{Q_0}}kQ \rightsquigarrow g: kQ/I \xrightarrow[/k^{Q_0}]{\sim} kQ/gI$

$$\begin{tikzcd} 0 \arrow[r, "U"'] & 1 \arrow[r, "V"'] & 2 \end{tikzcd} \rightsquigarrow I\in Gr(3, V\otimes U)$$

Conversely, any point $J\in Gr(3,V\otimes U)$ is a 2-sided ideal

$\mathrm{coh}\,(\mathbb{P}^2; \underbrace{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)}) \rightsquigarrow A = \mathrm{End}\, C \;) \rightsquigarrow I \mod \underbrace{\mathrm{Aut}_{k^{Q_0}} kQ}$

deform (L-VdB) $\qquad$ def. $\qquad$ def. $\qquad$ def.

$\mathcal{E}_i$ exc. $\rightsquigarrow \mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$
$\quad i = 1,2$

Def: $\mathcal{M}_{nc\mathbb{P}^2} := \left[Gr(3, V\otimes U) / \underbrace{\mathrm{Aut}_{k^{Q_0}} kQ}_{\substack{\wr\wr \\ GL(V)\times GL(U)}} \right]$

$$\begin{array}{ccl} M_{nc\mathbb{P}^2} & = & Gr(3, V\otimes U)^{ss} / SL(V)\times SL(U) \\ \cup & & \quad \cup \\ \mathring{M}_{nc\mathbb{P}^2} & = & Gr(3, V\otimes S)^{s} / SL(V)\times SL(U) \end{array}$$

$\leftarrow$ compact moduli of $nc\mathbb{P}^2$

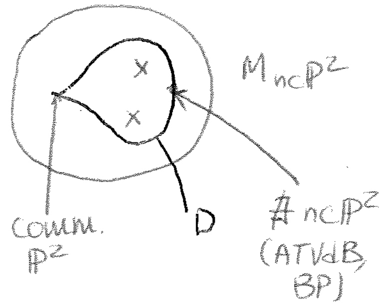
Thm (AOU)

1) $M_{nc\mathbb{P}^2} \simeq \mathbb{P}(6,9,12)\ (\simeq \mathbb{P}(1,3,2))$

2) $\mathring{M}_{nc\mathbb{P}^2}(k) \simeq \{ \mathrm{qgr}\, S \mid S : \mathrm{Skl.\ alg} \} /_{\mathrm{equiv.}} \underset{[\mathrm{St\text{-}VdB}]}{\simeq} \{ A \mid A : \mathrm{Skl}\ \mathbb{Z}\text{-alg} \} /_{\mathrm{isom}}$

3) $D = M \setminus \mathring{M} \simeq$ cusp. cubic curve

4) All points on D but 2 correspond to $nc\,\mathbb{P}^2$ whose curves are $\bigtriangleup$
(A-T-VdB, B-P)



5) $\exists$ birational construction

$\overline{M}_{1,2} \twoheadrightarrow M_{nc\mathbb{P}^2}$

coarse moduli space of stable curves $(g,n) = (1,2)$

§2 other del Pezzo's

$\mathbb{P}^1\times\mathbb{P}^1,\ \mathcal{O},\ \mathcal{O}(1,0),\ \mathcal{O}(1,1),\ \mathcal{O}(2,1)$ $\qquad$ FSEC

$$\circ \rightrightarrows \circ \rightrightarrows \circ \rightrightarrows \circ$$

2 relations

$$M_{nc\mathbb{P}^1\times\mathbb{P}^1} = Gr\left(2, (k^2)^{\otimes 3}\right)^{ss} / SL(2,k)^{\times 3}$$

dP of degree $d = 3, 2, 1$ ($\longleftrightarrow$ blow-up of $\mathbb{P}^2$ in $9-d$ points)

$\exists$ "nice" FSEC (Karpov–Nogin)
$\uparrow$
3-block

e.g. dP3

$P_1, \dots, P_6 \in \mathbb{P}^2$ $\quad\rightsquigarrow$

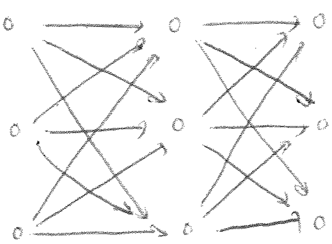
$f : X \longrightarrow \mathbb{P}^2$

$\underset{E_i}{\cup} \longrightarrow \underset{P_i}{\ni}$ $\qquad H = f^*(\mathcal{O}(1))$

$\mathcal{O}_X$	E_1	$H - E_4$
$\mathcal{O}_X(-H + E_1 + E_2 + E_3)$	E_2	$H - E_5$
$\mathcal{O}_X(-2H + E_1 + \dots + E_6)$	E_3	$H - E_6$

$(X, E_1, \dots, E_6)$ = marked cubic surface

$Q =$



$$\mathcal{M}_{dP3} = \left[(\mathbb{P}^2)^9 / \mathbb{G}_m^{18} \right]$$

$$M^{\Theta}_{dP3} = \left((\mathbb{P}^2)^9 \right)^{ss}(\Theta) / \mathbb{G}_m^{10} \quad \longleftarrow \text{8-dim. smooth variety}$$

toric var.

Θ = stab. param.

Prop: For $d = 3, 2, 1$, $\exists$ natural immersion

$$\begin{pmatrix} \text{moduli sp of marked} \\ \text{comm. dP of degree } d \end{pmatrix} \hookrightarrow M^{\Theta}_{ncdP,d} \quad \text{if } \Theta \text{ is generic}$$

d	dim	dim
1	8	10
2	6	9
3	4	8

Thm: $d = 3, 2, 1$ $\exists$ natural birational map

$$\{(Y, L_0, L_1, P_1, \dots, P_{9-d})\}/_{\text{iso}} \overset{\sim}{\dashrightarrow} M^{\Theta}_{ncdP\,d} \quad (\Theta \text{ generic})$$

ellip. curve $\qquad$ line bdls of degree 3

conjecture: $\exists$ action $\circlearrowleft$ of $W(E_{9-d}) \ltimes \mathbb{Z}^{9-d}$