Tony Pantev: Descent and equivalences in noncommutative geometry

Goal: descent etc. for dg categories without generators
= non-affine!

Bondal's philosophy: noncommutative space $X / \mathbb{C}$

$$\Updownarrow$$

dg / A_∞-category / $\mathbb{C}$, cocomplete, pretriangulated

= analogue of $D(\mathrm{Qcoh}\, X)$

Morphisms $f: X \to Y$

$$\Updownarrow$$

pairs of adjoint functors (dg functors) $D(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D(Y)$

$\ni f_*$ preserves colimits

(e.g. if $f^* \dashv f_* \dashv f^!$)

the most convenient ones are "nc affine" i.e. $\exists\, \xi \in D(X)$ perfect

$\ni$ $D(X) \cong A_\xi^{op}$-mod^{dg}, $A_\xi^{op} = \mathrm{End}(\xi)$

Goal: develop formalism of descent for noncommutative spaces, to handle Fourier-Mukai transforms, Morita equivalences etc without generators

dg is much richer than abelian in examples!

$\uparrow$

Tool: nc localization, from Kontsevich-Rosenberg

Def: $f: A \to B$ (of dg algebras) is a noncommutative localization morphism if $\mathrm{cone}\left(B \overset{L}{\underset{A}{\otimes}} B \to B\right) = 0$ in $B \otimes_A B^{op}$-mod

Example $X/\mathbb{C}$ separated of finite type, $Z \hookrightarrow X$ Zariski closed, $U = X \setminus Z \hookrightarrow X$ gives "many recollement"

$$\begin{tikzcd} D(U) \arrow[r, "j_*"] & D(X) \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "i_*"] & D_Z(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$$

corresponding to

$$U \xrightarrow{j} X \xrightarrow{i} X_Z$$

If we write $D(X) = A\text{-mod}^{\text{dg}}$

then U has a generator, and

$$U \hookrightarrow X$$

is incarnated by a localisation morphism

$$A \longrightarrow \mathrm{Hom}_{D(U)}(j^* A, j^* A)^{op}$$

<u>Construction</u> $f: A \to B$ nc localisation, get

$$C_{A,B} = \{ M \in A\text{-mod} \mid B \otimes_A M = 0 \}$$

- has all colimits
- may not have compact objects
  (in commutative case it does, Koszul complexes)
- only natural object is cone $(A \to B)$

<u>Example</u> $A = \mathbb{C}[x]$

$B = \mathbb{C}[x, (1/_{x-p})_{p \in S}]$

for any $S \subset \mathbb{A}^1$ [illegible]

<u>Properties</u>

1) $f: A \to B$ localisation

$\Rightarrow f^{op}$ is

2) $f: A \to B$, $f': A' \to B'$ localisations

$\Rightarrow A \otimes A' \longrightarrow \mathrm{cone}\left((A \otimes B') \oplus (A' \otimes B) \to B \otimes B'\right)(1)$

e.g. $U_1 \hookrightarrow X_1$, $U_2 \hookrightarrow X_2$

giving $(U_1 \times X_2) \cup (U_2 \times X_1) \subset X_1 \times X_2$

3) colimit-preserving functors $\mathrm{Fun}(C_{A,B}, C_{A',B'})$

~~correspond to~~

$$\begin{gathered} \mathrm{Fun}(C_{A,B}, C_{A',B'}) \\ \cong \\ C_{A \otimes A', \mathrm{cone}(\dots)} \end{gathered}$$

$= \{ M \in A \otimes A'\text{-mod} \mid M \otimes_A B = 0,\ B' \otimes_{A'} M = 0 \}$

$\rightsquigarrow$ equivalences are given by $\left\{ \begin{array}{l} M \in A \otimes A'\text{-mod} \\ N \in A' \otimes A\text{-mod} \end{array} \right.$ annihilated by B, B' on both sides

(= exactly the type of structure you want)

$$M \otimes_A N \simeq \mathrm{cone}\,(A' \to B') \in A' \otimes A'\text{-mod}$$
$$N \otimes_A M \simeq \mathrm{cone}\,(A \to B) \in A \otimes A\text{-mod}$$

= precisely the wrapped Fukaya category constructions

↳ analytic nongenerators

<u>Zariski descent</u>

X scheme, separated of finite type, $X = \bigcup_{i \in I} U_i$ Zariski open cover

for I a poset, $U_{i_1} \cap U_{i_2} = \bigcup_{j \leq i_1, i_2} U_j$ nerve of a Čech cover

This gives a small category $\mathcal{A}$,

$$\mathrm{Ob}(\mathcal{A}) = I$$

$$\mathrm{Hom}_{\mathcal{A}}(i,j) = \begin{cases} \Gamma(U_j, \mathcal{O}_X) & i \leq j \\ 0 & i \not\leq j \end{cases}$$

quasicoherence condition

Presheaves are $\mathcal{A}$-mod

Sheaves are $\mathcal{C}_{\mathcal{A},B} = \left\{ \mathcal{E} : \mathcal{A} \to \mathrm{Ch}(\mathbb{C}) \;\middle|\; \mathrm{cone}\left(\mathcal{O}(U_j) \otimes_{\mathcal{O}(U_i)} \mathcal{E}_i \to \mathcal{E}_j\right) \simeq 0 \right\}$

→ sheafification functor?

composition with inclusion should give

$$\mathcal{A}\text{-mod} \xrightarrow{T'} \mathcal{A}\text{-mod}$$ $T' \circ T' = T'$ = idempotent

with T' a coimpotent comonad

<u>Remark</u>: $T' : A\text{-mod}^{\circlearrowleft}$ equipped with $\varepsilon' : T' \to \mathrm{id}$

and if we have that $\forall M \in A\text{-mod}$

$$\mathrm{cone}\left(T' \circ T'(M) \to T'(M)\right) = 0$$

then it is canonically a ~~coimpotent~~ idempotent comonad.

→ combinatorial question! = representations of quivers

__Problem:__ ε' in Zariski descent setting?

__Example__ gluing two affines, $U_{1,2} = U_1 \cap U_2$ affine

$$\begin{tikzcd} & \mathcal{O}_1 \arrow[d] & & \mathcal{O}_2 \arrow[dl] \\ A := & \mathcal{O}_{1,2} & & \end{tikzcd}$$

$$A\text{-mod} = \left\{ \begin{tikzcd} M_1 \arrow[dr] & & M_2 \arrow[dl] \\ & M_{1,2} & \end{tikzcd} \;\middle|\; \begin{tikzcd} \mathcal{O}_{12} \otimes M_1 \arrow[dr] & \\ & \otimes M_{1,2} \\ \mathcal{O}_{1,2} \otimes M_2 \arrow[ur] & \end{tikzcd} \right\}$$

$= \otimes$, the map

now sheafification T' sends

actually modules appear

$$\left(\begin{tikzcd} M_1 \arrow[dr] & & M_2 \arrow[dl] \\ & M_{1,2} & \end{tikzcd} \right) \text{ to } \left(\begin{tikzcd} {\left[M_1 \underset{0}{\otimes} (\mathcal{O}_{1,2} \otimes M_2) \xrightarrow[1]{} M_{12}\right]} \arrow[dr] & & {\left[(\mathcal{O}_{1,2} \underset{0}{\otimes} M_1) \otimes M_2 \xrightarrow[1]{} M_{12}\right]} \arrow[dl] \\ & {\left[(\mathcal{O}_{12} \otimes M_1) \underset{0}{\otimes} (\mathcal{O}_{1,2} \otimes M_2) \xrightarrow[1]{} M_{12}\right]} & \end{tikzcd} \right)$$

$\rightsquigarrow$ $\varepsilon' : T' \to id$ via

$$0 \to M[-1] \to \operatorname{cone}(\varepsilon')(M) \to T'(M) \to 0$$

you need to add arrows of degree 1!

quasi-affine: replace $\Gamma(U, \mathcal{O}_X)$ by $\underline{R}\Gamma(U, \mathcal{O}_X)$?!