

Tony Pantev: Descent and equalizers in noncommutative geometry

Goal: descent etc. for dg categories without generators = non-affine!

Bondal's philosophy: noncommutative space  $X/\mathbb{C}$

↓

dg /  $A_\infty$ -category /  $\mathbb{C}$ , cocomplete, pretriangulated  
 = analogue of  $D(\text{Qcoh } X)$

Morphisms  $f: X \rightarrow Y$

↓

pairs of adjoint functors (dg functors)  $D(X) \begin{matrix} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{matrix} D(Y)$   
 $\approx f_*$  preserves colimits  
 (e.g. if  $f^{*-1} f_* \rightarrow f^*$ )

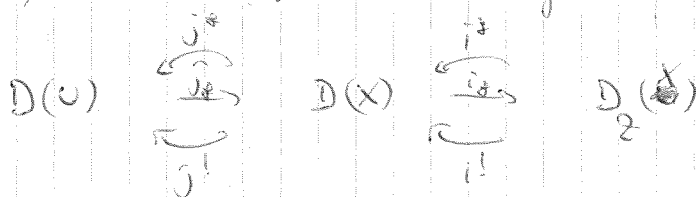
The most convenient one is 'no affine' i.e.  $\exists \mathcal{E} \in D(X)$  perfect  
 $\rightarrow D(X) \cong A_{\mathcal{E}}^{\text{op}} \text{-mod}$   $A_{\mathcal{E}}^{\text{op}} = \text{End}(\mathcal{E})$

Goal: develop formalism of descent for noncommutative spaces, to handle Fourier-Mukai transforms, Morita equivalences, etc without generators dg is much richer than abelian & examples!

Tool: nc localization, from Kontsevich-Rosenberg

Def.  $f: A \rightarrow B$  (of dg algebras) is a noncommutative localization morphism if  $\text{cone}(B \otimes_A^L B \rightarrow B) = 0$  in  $B \otimes_A B^{\text{op}} \text{-mod}$

Example  $X/\mathbb{C}$  separated of finite type,  $Z \hookrightarrow X$  Zariski closed,  $U = X \setminus Z \hookrightarrow X$  gives "strong recollament"



corresponds to



If we write  $D(X) = A\text{-mod}^{\text{af}}$

then  $U$  has a generator, and

$$U \cong X$$

is incarnated by a localization morphism

$$A \rightarrow \text{Hom}_{D(U)}(\mathcal{O}_X, \mathcal{O}_X)$$

Construction  $f: A \rightarrow B$  no localization, get

$$C_{A,B} = \{ M \in A\text{-mod} \mid B \otimes_A M = 0 \}$$

- has all colimits

- may not have compact objects

(if commutative case: it does, kernel complex)

- only natural object is cone  $(A \rightarrow B)$

Example  $A = \mathbb{C}[x]$

$$B = \mathbb{C}[x, (1/x)_s, s \in S]$$

for any  $S \subset A'$

Properties

1)  $f: A \rightarrow B$  localization

$\Rightarrow f^{\text{an}}$  is

2)  $f: A \rightarrow B, f': A' \rightarrow B'$  localizations

$$\Rightarrow A \otimes A' \rightarrow \text{cone}((A \otimes B') \otimes (A' \otimes B) \rightarrow B \otimes B') \quad (1)$$

e.g.  $U_1 \cong X_1, U_2 \cong X_2$

$$\text{giving } (U_1 \times X_2) \cup (U_2 \times X_1) \subset X_1 \times X_2$$

3) colimit-preserving functors  $\text{Fun}(C_{A,B}, C_{A',B'})$

~~is equivalent to~~

$$C_{A \otimes A', \text{cone}(\dots)}$$

$$= \{ M \in A \otimes A'\text{-mod} \mid M \otimes_A B = 0, B' \otimes_{A'} M = 0 \}$$

$\rightarrow$  equivalences are given by

$$\begin{cases} M \in A \otimes A'\text{-mod} \\ N \in A' \otimes A\text{-mod} \end{cases}$$

$\cong$  annihilated by

$B, B'$  on both sides

(= exactly the type of structure you want)

$$M \otimes_A N \cong \text{con} (A' \rightarrow B') \in A' \otimes A' \text{-mod}$$

$$N \otimes_A M \cong \text{con} (A \rightarrow B) \in A \otimes A \text{-mod}$$

= precisely the wrapped Fukaya category construction  
 is analytic no generators

Zariski descent

X scheme separated of finite type,  $X = \bigcup_{i \in I} U_i$  Zariski open cover

for  $i \neq j$  part,  $U_i \cap U_j = \bigcup_{k \in I, k \neq i, j} U_k$  nerve of  $\check{C}$ ed cover

This gives a small category  $\mathcal{A}$ ,

$$\text{Ob}(\mathcal{A}) = I$$

$$\text{Hom}_{\mathcal{A}}(i, j) = \begin{cases} \Gamma(U_i, \mathcal{O}_X) & i = j \\ 0 & i \neq j \end{cases}$$

quiver  
 conditions  
 $\downarrow$

Presheaves are  $\mathcal{A}$ -mod

$$\text{Sheaves are } \mathcal{C}_{\mathcal{A}} B = \left\{ \begin{array}{l} \mathcal{C} : \mathcal{A} \rightarrow \mathcal{C}(\mathcal{C}) \\ \text{con} (\mathcal{O}(U_i) \otimes_{\mathcal{C}(U_i)} \mathcal{C}(U_j) \rightarrow \mathcal{C}(U_i \cap U_j)) = 0 \end{array} \right\}$$

→ sheafification functor?

composition with inclusions should give

$$A \text{-mod} \xrightarrow{T} \mathcal{A} \text{-mod} \xrightarrow{T'} \mathcal{A} \text{-mod} \quad T' \circ T = T' = \text{idempotent}$$

with  $T'$  a coisotropic comonad

Remark:  $T' : A \text{-mod} \mathcal{G}$  equipped with  $\epsilon : T' \rightarrow \text{id}$

and if we have that  $\forall M \in A \text{-mod}$

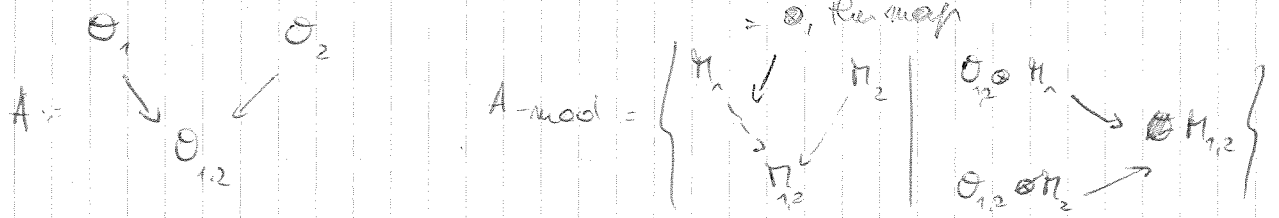
$$\text{con} (T' \circ T'(M) \rightarrow T'(M)) = 0$$

then it is canonically a coisotropic idempotent comonad.

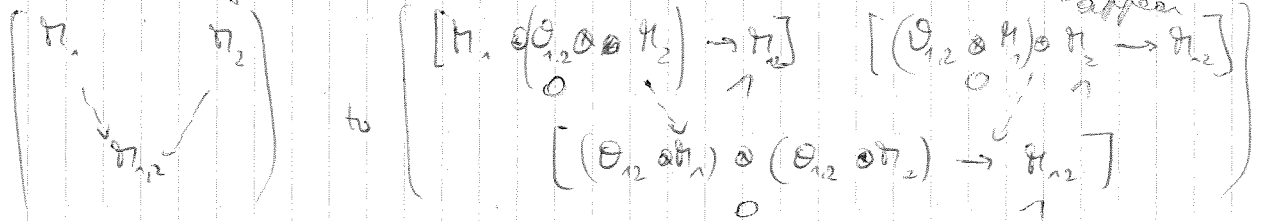
→ combinatorial question! = representations of quivers

Problem:  $\varepsilon$  of  $\mathcal{T}$  Zariski descent setting?

Example gluing two affines,  $U_{1,2} = U_1 \cap U_2$  affine  
 $= \mathcal{O}_1$  Rem map



new sheafification  $\tau$  sends



actually by modules appear

$\varepsilon: \mathcal{T} \rightarrow \mathcal{A}$  via

$$0 \rightarrow \mathcal{T}[-1] \rightarrow \text{coker}(\varepsilon)(\mathcal{T}) \rightarrow \mathcal{T}'(\mathcal{T}) \rightarrow 0$$

you need to add amount of degree 1!

quasi-affine: replace  $\mathcal{T}(U, \mathcal{O}_X)$  by  $\mathcal{R}\mathcal{T}(U, \mathcal{O}_X)$ ?!/