Polishchuk | Moduli of A_∞ structures

(on a fixed associative algebra)

There will be geometry in this.

<u>Recall</u>: A $\mathbb{Z}$ gr vec sp

"not Chow gp"

Hoch cochains $CH^{s+t}(A)_t = \mathrm{Hom}(A^{\otimes s}, A)_t$

unusual grading; more natural. "total degree" internal grading

Given $c \in CH$, have coder of $Bar(A)$

$$D_c : Bar(A) \circlearrowleft$$
$$\| $$
$$\bigoplus_{i \geq 1} T^i(A[1]) \quad \text{free coalg gen by } A[1] \text{ (no unit)}$$

and $[D_c, D_{c'}]$ Gerstenhaber bracket

gr comm. $D_{[c,c']}$.

An A_∞ str on A is

$$(m_1, m_2, \cdots) \in CH^2(A)$$

each in tot deg 2

s.t.

$$[m, m] = 0$$

Rmk: Assume 2 invertible for simplicity.

$[m_1, m_1] = 0$
$[m_1, m_2] = 0,$
$[m_1, m_3] + [m_2, m_2] + [m_3, m_1] = 0$

Have notion of equivce: Gauge equivce.

$$(f_2, f_3, \cdots) \in CH^1, \quad f_2 : A^{\otimes 2} \to A[-1]$$
$$f_3 : A^{\otimes 3} \to A[-2]$$

$\rightsquigarrow \quad \alpha_f : Bar(A) \circlearrowleft$ automorphism

autom act on coderivs:

$$\alpha_f D_m \alpha_{f^{-1}} = D_{f * m}.$$

So $\{f\} \curvearrowright \{m\}$.

Gauge equiv. $= \mathfrak{g}$ $\quad$ A_∞ str.

Fix fin-dim gr assoc alg A. $\quad \dim\left(\bigoplus_i A_i\right) < \infty$

(ie, m_2 fixed)

($m_1 = 0$)

Consider minimal ($\overset{\text{ie}}{m_1 = 0}$) A_∞ str on A w/ given m_2, up to gauge

<u>Rmk</u> Gauge action <u>won't</u> change m_2.

$\rightsquigarrow$ functor $\text{Comm} \longrightarrow \text{Sets}$

(R-linear A_∞ str on $A \otimes R$)

<u>Thm</u> Assume $HH^1(A)_{\underbrace{<0}_{\substack{\text{negative}\\ \text{int. degree}}}} = 0$.

Then functor is representable by an affine scheme.

"moduli of A_∞ str on A"

If $HH^2(A,A)_{<0}$ is fin-dim, $\underbrace{\text{it's}}_{\text{the scheme}}$ of finite type.

ie, any A_∞ str is determined by <u>finitely</u> many m_n.

<u>Rmk</u> Not considering whole ∞-gpd str.
Just modding out by Gauge equiv.

Ex 1 C genus g ~~smth~~ reduced, connected proj curve (g = arithm genus) ie, $h^1(\mathcal{O}) = g$

$p_1, \dots, p_g$ points, all smth pts. $\quad g \geq 1$

Consider object $\quad G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_g}$

Assume $p_1 + \cdots + p_g = D$ is non-special, so

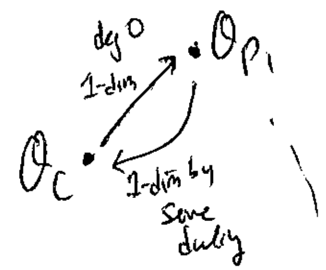
$$\text{(i)} \quad h^1(\mathcal{O}(p_1 + \cdots + p_g)) = 0.$$

Assume (ii) $\mathcal{O}(p_1 + \cdots + p_g)$ ample. ie, on each irred compon, $\exists$ a marked pt.

G generates Perf(C).

(i) $\Rightarrow$ $\mathrm{Ext}^\bullet(G, G)$ indep of C, p_i.

Quiver descrip:



Composition $\mathcal{O}_{p_1}$ $\mathcal{O}_{p_g}$

gives ext generator of deg 1.

$\Rightarrow$ g loops, but $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = H^1(\mathcal{O})$ has dim g. $\leftarrow g \geq 1$; $g = 0$, no points.

$\mathcal{O}_C$ g loops

Need choice of tangt vectors:

$$\mathrm{Ext}^{*}(G,G) \overset{\text{isom}}{\cong} E_g$$

depends on choice of non-0 tang vecs

(Can also be described as path alg mod cubic relns)

Passing to dg enhance, get A_∞ str on E_g.

Homological perturbation: get canonical minimal A_∞ str on $\mathrm{Ext}^{*}(G,G)$.

up to gauge.

$\rightsquigarrow$ get map from "moduli of curves" to "moduli of A_∞ str"

(can prove works in families).

Thm $\widetilde{\mathcal{U}}^{ns}_{g,g} \longrightarrow \{A_\infty \text{ str on } E_g\}/\text{gauge}$

"$\sim$" means choice of tang vecs.

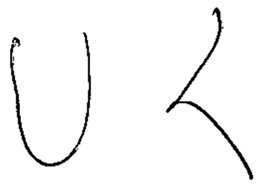
is an isomorphism.

Special point on $\widetilde{U}^{ns}_{g,g}$: has action of multip gp, so extra symm.

$\ni$

C_g^{cusp} eg $g=1$ cuspidal cubic

$\shortparallel$



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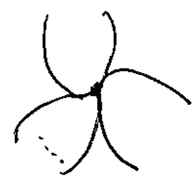


tangentially unioned @ cusp

$\updownarrow$

trivial A_∞ str: all higher products are 0.

Can show $\widetilde{U}^{ns}_{g,g}$ <u>affine</u>; both $\widetilde{U}^{ns}_{g,g}$ and $\{A_\infty\}$ are like cones, and only need to compare deform thy of C_g^{cusp} and (m)^{triv}.

That results in proof.

Ex 2 Consider A_∞ str on

$$\begin{tikzcd} \overset{+1}{\circlearrowleft} \underset{\text{1-dim}}{\circ} \arrow[r, bend left, "V"] & \underset{\text{1-dim}}{\circ} \overset{+1}{\circlearrowright} \arrow[l, bend left, "+1", "V^*"'] \end{tikzcd}$$

compositions are pairing on V, V^*.

Appears taking two ~~...~~ bundles on ell curve E, s.t. $\hom^{\neq 0}(L_0, L_1) \overset{\text{ext}^1}{=} 0$ and $\mathrm{Ext}^1(L_0, L_1)$

simple vector

$\hom(L_0, L_1) =: V$,

get A_∞ str on $\rightsquigarrow$ $\mathrm{Ext}^*(L_0 \oplus L_1, L_0 \oplus L_1)$

take degenerations to more singular curves

To each A_∞ str, ~~associate~~ get soln to some "associative Yang-Baxter" equation

Given a 1-CY category, 1-spherical objects E, F, (eg. simple vect bds)

st $\cdot \hom^0(E, F) \neq 0$

$\cdot \hom^{\neq 0}(E F) = 0$

then $\mathrm{Ext}^*(E \oplus F, E \oplus F)$ is algebra.

But $\exists$ 1-param family $E(x)$, $F(y)$

s.t. $E(x), E(x')$ mutually $\perp$, so $\hom(E(x), E(x')) = 0$

$\qquad \qquad \| \quad \hom(E(x'), E(x))$

Take family st. one has identification

$\hom(E(x), F(y)) \cong V$.

⑧

$\leadsto$ Triple Massey product

$$E(x_1) \xrightarrow[V]{} F(y_1) \xrightarrow[V^*]{\overset{\mathrm{ext}^1(F(y_1),\,E(x_2))}{\overset{\cup}{+1}}} E(x_2) \xrightarrow[V]{} F(y_2)$$

if $x_1 \neq x_2$, $y_1 \neq y_2$, get hom $=0$, hence well-def. Massey product:

$$V \otimes V^* \otimes V \longrightarrow V$$

$\Big\}$ dualize

$$V \otimes V \longrightarrow V \otimes V$$

call map r, for "r matrix", depending on $\begin{matrix} x_1, x_2 \\ y_1, y_2 \end{matrix}$

$$r^{x_1, x_2}_{y_1, y_2} \in \mathrm{End}(V)^{\otimes 2}$$

Theorem is r satisfies an assoc. Y-B equation: Omit x, y for now.

<u>Assoc Y-B eqn</u>:

$$r^{12} r^{13} - r^{23} r^{12} + r^{13} r^{23} = 0$$

in $\mathrm{End}(V)^{\otimes 3}$

$r^{12} = r \otimes 1$, etc.

Really, have x_1, x_2, x_3, y_i,

$$r^{12} = \left(r^{x_2 x_1}_{y_2 y_3}\right)^{12}, \quad \text{etc.}$$

Now, if you take $[\,,\,]$ not $\cdot$, get <u>classical</u> YB eqn.

From assoc YB, get classical YB if you collapse y_i, or x_i, to each other and take some limit.

Even w/o 1-param object, we find variables x, y using twisted objects over A_∞ cat. Get "formal solns" co

$$r \in \mathrm{End}(V)^{\otimes 2}[[x_1, x_2, y_1, y_2]]\underbrace{[(x_1-x_2)^{-1}, (y_1-y_2)^{-1}]}_{\text{to keep } x_i, y_i \text{ separate}}$$

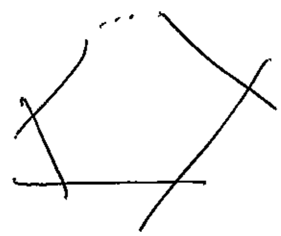
<u>Thm</u> (w/ Lekili) get Bam of moduli spaces.

$$\{\overline{A_\infty}^{\text{cyclic}} \text{ str}\} \cong \left\{\begin{matrix}\text{solns to}\\ \text{Assoc YB}\end{matrix}\right\} \Big/ \text{equiv}$$

<u>Application</u>: (Fuk for surfaces)

exact Fuk cat of punctured torus

<u>Thm</u> Any two simple vec bdls on



cycle of $\mathbb{P}^1$,
a degen. of
ell curve

are connected by
sequence of spherical twists.

$$E_1 \xmapsto[\substack{\text{sph}\\ \text{twist}\\ \text{functor}}]{} E_2 \longmapsto \cdots \longmapsto E_n.$$

Point: given V simple, $\mathcal{O}_p$ p smooth point, see YB equ
arises from Fuk cat, related by Dehn twists

<u>Rmk</u> $\mathcal{O}_p \longmapsto \mathcal{O}_q$
by ~~...~~ spherical twist.