Positselski

$\mathcal{C}$ Ab cat

$$D(C) = \mathrm{Hot}(C)/\mathrm{Acyc}(C)$$ unbdd dvd cat

$$D^{co}(C) = \mathrm{Hot}(C)/\mathrm{Acyc}^{co}(C)$$ defined if $\oplus$ exact in $\mathcal{C}$

"coderived"

$$= \langle \mathrm{Tot}(K^\bullet \to L^\bullet \to M^\bullet \text{ ses}) \rangle_{\oplus}$$ closed under $\oplus$

$$D^{ctr} = \mathrm{Hot}(C)/\mathrm{Acyc}^{ctr}(C)$$ defined if Π exact in $\mathcal{C}$.

"contra"

"contraderived"

$$= \langle \mathrm{Tot}(K^\bullet \to L^\bullet \to M^\bullet \text{ ses}) \rangle_{\Pi}$$ closed under Π

Prop - C • enough inj

• $\bigoplus_{\text{countable}}$ injectives has finite inj dim

$\Rightarrow$ $D^{co}(C) \simeq \mathrm{Hot}(C_{inj})$

- C • enough proj

• $\prod_{\text{countable}}$ projectives has finite proj dim

$\Rightarrow$ $D^{ctr}(C) \simeq \mathrm{Hot}(C_{proj})$

Prop $\mathcal{C}$ finite homol dim

$$\Longrightarrow \quad D^{co} \underset{\text{if } \oplus \text{ exact}}{\simeq} D \underset{\text{if } \Pi \text{ exact}}{\simeq} D^{ctr}$$

Given $\mathcal{C}$ coun coass ~~coass~~ coalg / k fld.

ie, data of ν

left $\mathcal{C}$ comod M is

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[dr, "id"'] & \mathcal{C}\otimes M \arrow[r, shift left, "1\otimes\nu"] \arrow[r, shift right, "\nu\otimes 1"'] \arrow[d, "\varepsilon\otimes 1"] & \mathcal{C}\otimes\mathcal{C}\otimes M \\ & M & \end{tikzcd}$$

$$\mathcal{C} \xrightarrow{\varepsilon} k \qquad \mathcal{C} \xrightarrow{\mu} \mathcal{C}\otimes\mathcal{C}$$

left $\mathcal{C}$ contramod P is

$$\begin{tikzcd} \mathrm{hom}(\mathcal{C}, \mathrm{hom}(\mathcal{C}, P)) \arrow[r, "\pi_*"] \arrow[d, phantom, "\cong"] & \mathrm{hom}_k(\mathcal{C}, P) \arrow[r, "\pi"] & P \\ \mathrm{hom}(\mathcal{C}\otimes_k\mathcal{C}, P) \arrow[ur, "\mu^*"] & P \arrow[u, "\varepsilon^*"] \arrow[ur, "id"'] & \end{tikzcd}$$

ie, data of π

	mod	comod / contramod

D

mod:

$D(A\text{-Mod}) \simeq D(A\text{-Mod})$

A assoc ring

ex Vir^{+} alg, Vir^{-} coalg, c central charge

$D^{\text{semico}}(\mathcal{O}_c(Vir)) \simeq D^{\text{semictr}}_{-26+c}(\mathcal{O}_c^{ctr}(Vir))$

Vir = Virasoro algebra.

comod / contramod:

Mathis-Greenlees-May duality for coalgebras

$D(\mathcal{C}\text{-comod}) \simeq D(\mathcal{D}\text{-contramod})$

$\mathcal{C}, \mathcal{D}$ coass / k field,

$\mathcal{C}$ left cocoherent, $\mathcal{D}$ right cocoherent

W "dedualizing" complex ${}_{\mathcal{C}}B_{\mathcal{D}}$

$$M \longmapsto \mathrm{Rhom}_{\mathcal{C}}(B, M)$$
$$B \overset{\mathbb{L}}{\underset{\mathcal{D}}{\otimes}} P \longleftarrow P$$

more generally if S alg obj in $\mathcal{C}$-comod-$\mathcal{C}$, wrt $\square_{\mathcal{C}}$, then $D^{sco}(S\text{comod}) \simeq D^{sctr}(S\text{sctr})$

via
$$M \longmapsto \mathbb{R}\mathrm{hom}_S(S, M)$$
$$S \overset{\mathbb{L}}{\underset{S}{\otimes}} P \longleftarrow P$$

D^{co} / D^{ctr}

mod:

Covariant Serre-Grothendieck duality (Iyengar-Krause)

$D^{co}(A\text{mod}) \simeq D^{ctr}(B\text{mod})$

A, B assoc

A left coherent, B right coherent,

W a dualizing complex ${}_A\mathcal{D}_B$.

$$M \longmapsto \mathbb{R}\mathrm{hom}_A(\mathcal{D}, M)$$
$$\mathcal{D} \overset{\mathbb{L}}{\underset{B}{\otimes}} P \longleftarrow {}_B P$$

comod / contramod:

Co-contra Correspondence!

$D^{co}(\mathcal{C}\text{-comod}) \simeq D^{contra}(\mathcal{C}\text{contramod})$

$\forall\ \mathcal{C}$ coass coun. coalg / k field

$$M^{\bullet} \longmapsto \mathrm{Rhom}_{\mathcal{C}\text{comod}}(\mathcal{C}, M)$$
$$\mathcal{C} \overset{\mathbb{L}}{\underset{\mathcal{C}}{\odot}} P^{\bullet} \longleftarrow P^{\bullet}$$

contramod version of $\otimes$. Won't define.

Ex Ab gps

M torsion iff $\mathbb{Q} \otimes_{\mathbb{Z}} M = 0$

P reduced cotorsion iff $\underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, P) = 0}_{\text{reduced}} \underbrace{=}_{\text{cotorsn}} \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, P)$

$\{\text{torsion } M\} \subset Ab$

└ Serre subcat — ie, closed under subs, quots, exts (extensions), $\oplus$. abelian cat.

$\{\text{reduced cotors}\} \subset Ab$

└ abelian subcat — ie, closed under (co)ker, extensions, infinite $\prod$.

subgps need NOT be reduced cotor, but any hom btw red. cot. have red. cot. (co)kernel.

Thm $\mathcal{D}(Ab_{tors}) \simeq \mathcal{D}(Ab_{red\,cot})$

Pf Two eqvlnces: ① $\mathcal{D}(Ab_{tors}) \simeq \mathrm{Hot}(Ab^{inj}_{tors}) \simeq \mathrm{Hot}(Ab^{proj}_{redcot}) \simeq \mathcal{D}(Ab_{redcot})$

enough inj in Ab_{tors}
Ab_{tors} has hom dim $= 1 < \infty$

enough proj in Ab_{redcot}
homol. dim $(Ab_{redcot}) = 1 < \infty$

$Ab^{inj}_{tors} \simeq Ab^{proj}_{redcot}$ (as additive cats)

$M \longmapsto \mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M)$

$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} P \longleftarrow P$

"profinite" or "adic" completions not important

Pf ② $\mathcal{D}(Ab_{tors}) \simeq \mathcal{D}(Ab)/\mathcal{D}(\mathbb{Q}\text{-vec}) \simeq \mathcal{D}(Ab_{redcot})$

two semiorthog decomp:

$D(Ab) \supset D(Ab_{tors}) = {}^{\perp}D(\mathbb{Q}\text{vect})$

$D(\mathbb{Q}\text{-vect})^{\perp} = D(Ab_{redcot})$

R Noeth comm ring, $I \subset R$ ideal

$$R\text{-mod} \supset R\text{-mod}_{I\text{tors}} = \{ M \mid R[s^{-1}] \underset{R}{\otimes} M = 0 \ \ \forall s \in I \}$$

arbit. large

$$R\text{mod} \supset R\text{mod}_{I\text{ctm}} = \left\{ P \;\middle|\; \begin{array}{l} \hom_R(R[s^{-1}], P) = 0 \ \ \forall s \in I \\ \operatorname{Ext}^1_R(R[s^{-1}], P) = 0 \end{array} \right\}$$

and $R\text{-mod}_{I\text{tors}}$ Serre subcat, abelian.

$R\text{mod}_{I\text{ctm}}$ abelian subcat

$\Gamma_I : R\text{-mod} \longrightarrow R\text{-mod}_{I\text{tors}}$ right adj to inclusion.

$$\rightsquigarrow \quad R^b\Gamma_I : D^b(R) \longrightarrow D^b(R\text{mod}_{I\text{tors}})$$

$\mathcal{B} := R\Gamma_I(R)$ "dedualizing"

$D := R\Gamma_I(\mathcal{D}_R)$ $\mathcal{D}_R$ dualizing cplx of R-mod.

D "t-dualizing" cpx.

Idea: (R, I) is

- "ring along R/I"
- "coalg in transversal dxn"

Rmk orthog. decomp via

$$\mathcal{D}_{I\text{tors}}(R\text{mod}) \simeq \mathcal{D}_{I\text{comp}}(R\text{mod})$$

H° is Itors $\qquad$ H° is I comp

where common orthogonal is sheaves supported on $\operatorname{Spec} R \setminus \operatorname{Spec}(R/I)$

i.e. supp on complement.