BOUNDED DERIVED CATEGORIES, NORMALISATIONS & DUALITY

G. Stevenson

j/w John Greenlees

$$\begin{array}{l} R \rightsquigarrow D^b(\mathrm{mod}\,R) \\ \downarrow \qquad\quad \cup\!\mathrm{l} \\ k \qquad D^{perf}(R) \end{array} \Bigg] \rightsquigarrow D_{sg}(R) = \frac{D^b(\mathrm{mod}\,R)}{D^{perf}(R)}$$

$$D_{cosg}(R) = \frac{D^b(\mathrm{mod}\,R)}{\mathrm{thick}(k)}$$

Goal: What is the analogue of $D^b(\mathrm{mod}\,R)$ if R is some more complicated homotopical gadget?

$R \longrightarrow k$ ← field

dga, ring spectrum

Homotopical "commutative" algebra:

Regular rings/maps:

Def: $g: S \longrightarrow R$ is relatively g-regular if $R \in D(S)^c$. (reg. from now on)

We say $R \longrightarrow k$ is regular if the map is, i.e. $k \in D(R)^c$.

- Ex: (R, m, k) if R is regular
  - If H^*R is (noeth.) and regular so is R
  - G finite p-group, $R = C^*(BG, k)$ is regular.

Gorenstein

Def: A morphism $g: S \longrightarrow R$ is relatively Gorenstein if $\mathrm{Hom}_S(R, S) \cong \Sigma^{G_R} R$, $G_R \in \mathbb{Z}$.

We say R is Gorenstein if $\mathrm{Hom}_R(k, R) \cong \Sigma^G k$

- Ex: G finite group, $C^*(BG, \mathbb{F}_p)$ is Gorenstein (DGI)

Idea: If R is a honest fp k-algebra, $k[z] \xrightarrow{\pi} R$ s.t. R finite $k[z]$-module, one can define:

$D^b(\mathrm{mod}\,R) = \{X \in D(R) \,/\, \pi_* X \in D^{perf}(k[z])\}$

A normalisation of R is a map $q: S \to R$ s.t. q is rel. regular + S is regular, i.e. $k, R \in D(S)^c$.

We define the q-bounded derived category

$$D^{q\text{-}b}(R) = \{ X \in D(R) \mid q_* X \in D(S)^c \}$$

- Remark: $k, R \in D^{q\text{-}b}(R)$ $\rightsquigarrow$ so can define cosg, sg relative to q.

- Ex: R regular, $R \xrightarrow{i} R$

  - $R = C^*(BG, \mathbb{F}_p)$, G finite group

    Choose faithful representation $G \to U(n)$ and consider

    $C^*(BU(n); \mathbb{F}_p) \to C^*(BG; \mathbb{F}_p)$ is a normalisation

Start with a normalization $q: S \to R$. We can take the cofibre $Q = R \otimes_S k$, $S \xrightarrow{q} R \xrightarrow{p} Q$

Set $\mathcal{E} = \mathrm{Hom}_R(k,k)$, similarly $S \rightsquigarrow \mathcal{F}$, $Q \rightsquigarrow \mathcal{D}$
$\mathcal{D} = \mathrm{Hom}_Q(k,k)$

Set

$$\begin{array}{c} S \xrightarrow{q} R \xrightarrow{p} Q \\ \mathcal{F} \xleftarrow{j} \mathcal{E} \xleftarrow{i} \mathcal{D} \end{array}$$

Lemma: $\mathcal{F} \xleftarrow{\cancel{j}} \mathcal{E} \xleftarrow{i} \mathcal{D}$ is also a cofibre seq. i.e. $\mathcal{F} \simeq \mathcal{E} \otimes_{\mathcal{D}} k$

Say $\begin{array}{c} S \xrightarrow{q} R \xrightarrow{p} Q \\ \mathcal{F} \xleftarrow{j} \mathcal{E} \xleftarrow{i} \mathcal{D} \end{array}$ is a symmetric Gorenstein ~~complex~~ context (SGC) if in these sequences 6+4 Gorenstein, 2+4 regular

i.e. all rings are Gorenstein (6), 4's all rings are rel. reg. Gor. + $S, \mathcal{D}$ are regular

Thm (Greenlees–S):

If $q: S \to R$ is s.t. q-rel. Gorenstein + normalization (rel reg + S reg) + S Gor

(+ one of $\mathcal{F}, \mathcal{E}, \mathcal{D}$ is Gorenstein) then we get a SGC.

We can consider the functors

$$E: D(R) \xrightarrow{\mathrm{Hom}_R(k,-)} D(\mathcal{E}) \xrightarrow{\overline{E}} D(\hat{R})$$

where $\hat{R} = \mathrm{Hom}_{\mathcal{E}}(k,k)$ - the dc-completion of R ($\exists$ map $R \to \hat{R}$)

Thm (GS)

Suppose we have a SGC + $R, S, \mathcal{E}, D$ are complete. Then E and $\bar{E}$ restrict to an equiv.

$$D^{q\text{-}b}(R) \xrightarrow{\sim} D^{q\text{-}b}(\mathcal{E})$$

which interchanges D^{perf} and $\mathrm{thick}(k)$.

$\Rightarrow D^{q}_{sg}(R) \simeq D^{i}_{cosg}(\mathcal{E})$ + dually

Thm For such SGC's $D^{q\text{-}b}$, $D^{i\text{-}b}$ are independent of q or i.

Ex: • R regular (eg SV), we can take $R \xrightarrow{1} R \longrightarrow k$ and

$\mathcal{E} \longleftarrow \mathcal{E} \longleftarrow k$

(eg ΛV)

we can recover usual Koszul duality.

• G finite p-group $\Rightarrow C^*(BG, \mathbb{F}_p)$ is regular

$D_{cosg}(C^*(BG, \mathbb{F}_p)) \cong D_{sg}(\mathcal{E}) = \underline{\mathrm{mod}}\ kG$

$\mathcal{E} \cong C_*(\Omega BG) = kG$

altern. group on 4 letters

• $G = A_4$, $k = \mathbb{F}_2$

$A_4 \longrightarrow SO(3)$

$C^*(BSO(3)) \longrightarrow C^*(BA_4)$ is a normalization