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Bloch conductor formula + Matrix fact.

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Want to prove Bloch conductor formula:

Today: partial results. Will use ℓ-adic realization of non-comm spaces.

$S = \operatorname{Spec} A$ A henselian ~~Noetherian~~ DVR. (complete, eg)

$A/m = k$ perfect field.

$K = \operatorname{Frac}(A)$ fraction field.

π a uniformizer.

Fix $X \xrightarrow{p} S$ proper map of schemes, assume · X regular

· X_K generic fiber smooth over K

Conj (Bloch)

$$\chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) = \deg[\Delta_X \cdot \Delta_X] + Sw(X_K)$$

χ in ℓ-adic cohom

geometric special fiber

geom generic fiber

Swan conductor

- ℓ prime to char
- $\chi(-)$ in ℓ-adic cohom
- $[\Delta_X \cdot \Delta_X]$ a 0-cycle on X_k, ways to define degree using K thy. Complicated, so omitted. But degree has K thy interpretation:

$\Omega^1_{X/S}$ relative Kähler forms, coh. sheaf on X

$$\underline{\mathbb{L}\Lambda^{n+1}\Omega^1_{X/S}} \qquad n = \text{relative dimension} = \dim X/S.$$

p not smth $\Rightarrow$ lives in $K_0(\mathrm{Coh}(X)$ supp in fiber$)$

ie, $\in G_0(X_k)$.

By p, push to $G_0(k) = \mathbb{Z}$.

Literature: $\deg[\Delta_X \cdot \Delta_X] \simeq p_*(\mathbb{L}\Lambda^{n+1}\Omega^1_{X/S})$

Tor not finite since p not smth, so χ not defined.
Tor becomes 2-periodic after a while, so can "normalize", so
$[\Delta_X \cdot \Delta_X]$ is morally self-$\cap$ of Δ_X.

- $\mathrm{Sw}(X_K)$ says something (arithmetic) of wild ramification. 0 of family is tame.
  Monodromy unipotent $\Rightarrow \mathrm{Sw} = 0$.

$\|$

Swan conductor of $H^*(X_{\bar{K}}, \mathbb{Q}_\ell) \circlearrowleft \mathrm{Gal}(K^{sp}/K)$

- X smth: conjecture reads: $0=0$

Known cases:

ie, no generic fiber. Factors thgh a pt.

ⓐ Assume p factors
$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "p"] & \operatorname{Spec} k \arrow[r] & S \end{tikzcd}$$

then $Sw = 0$

get Gauss-Bonnet formula:
$$\begin{array}{ccc} \chi(X_{\bar k}) & = & \deg[\Delta_X \cdot \Delta_X] \\ \cancel{\|} & & \| \\ & \sum(-1)^{p+q} \dim\left(H^p(X, \Omega^q_X)\right) & \end{array}$$

ⓑ If A equicharacteristic, $X_{\bar k}$ has isolated singularity: Deligne.

"Milnor-Deligne formula" SGA 7.

Char p analogue of Milnor formula for Milnor #.

↑ Swan conductor is extra term in char p, as opposed to char 0 case.

ⓒ k char 0, then Conj true: No Sw again. $A = \Bbbk[t]$

$$\deg[\Delta_X \cdot \Delta_X] = \chi\left(\underline{DR(X,f)}\right)$$

"twisted" deRh complex, $\underset{\overset{\|}{t}}{f} : X \longrightarrow \mathbb{A}^1_k$, twisted by f.

Kapranov: $\chi(X_{\bar k}) - \chi(X_{\bar K}) = \chi(DR(X,f))$.

ⓓ Kato-Saito. If $(X_k)_{\text{reduced}} \hookrightarrow X$ is a simple nc divisor.

$\Longrightarrow$ general case if one assumes strong resolution of singularities

② proven via trace formula in log geometry, log cohomology.

$$\chi(\text{log cohom}) \overset{\text{show}}{=} \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}).$$

We do same by replacing log geom by non-comm geom.

Monodromy always quasi-unipotent. Can base change to unipotent monodromy by finite base change.

But Sw depends on X, not just base change! This is the hard part.

Case ⓪: Relative dimension 1 (Bloch). True here, too.

Non Comm Geom:

Defn A nc space over ring A is a dg category over A. $\mathrm{ncspace}/_A := \mathrm{dgcat}/_A$.

∃ symm ⊗ ∞-cat of nc spaces over A; call it Sch^{nc}_A "non-comm schemes"

where

ob = small $\mathrm{dgCat}/_A$

hom = bimodules $T \dashrightarrow T'$ are $T \otimes (T')^{op}$ dg-mod E

st $E(t,-)$ is compact in $(T')^{op}\mathrm{Mod}$ $\forall t \in T$.

generally objects are pullbacks of schemes

By $T \otimes (T')^{op}$, mean $T \overset{L}{\underset{A}{\otimes}} (T')^{op}$.

∃ particular nc space of interest:

Given $X \xrightarrow{p} S$, consider π uniformizer (fxn on S, as fxn on X)

$$MF(X,\pi) = mF \text{ on } X \text{ for fxn } \pi.$$

$$\cap$$

$$Sch^{nc}_A$$ for now, 2-periodic dg cat, ignore periodicity.

Claim: a suitable l-adic theory for non-comm spaces exists.

Further, $\widehat{} H^\circ(-,\mathbb{Q}_\ell)$

$$\chi\left(H^\circ(MF(X,\pi),\mathbb{Q}_\ell)\right) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}})$$

Now try to generalize χ formula from before: $\chi(\ ) = [\Delta_X \cdot \Delta_X]$.

If non-comm space is smooth proper, have trace formulas in general.

NOT proper, but need some *finiteness* property to apply this.

Then, interpret terms in trace formula as $[\Delta_X \cdot \Delta_X]$ and Sw.

More about $MF(X,\pi)$. *NOT* a nice space over A:

1) Concentrated on $\mathrm{Sing}(X \xrightarrow{p} S)$, on special fiber.
It's *k-linear*, not A-linear naturally. So not

2) Not even nice over k, *NOT proper over k*, as hom not fin-dim (2-periodic)

⚠ NOT $k[u,u^{-1}]$-linear for $|u|=2$, either,
unless A is equicharacteristic.

Unlike $\mathbb{C}$ case, where MF is $k[u,u^{-1}]$ linear. (Ring has mixed characteristic)

Fact: $\exists$ E_2-algebra B (over A) s.t. • $H^*(B) \cong k[u,u^{-1}]$

• B acts on $MF(X,\pi)$

$$B := R\mathrm{End}_{\underset{\sim}{k \overset{L}{\otimes}_A k}}(k)[u^{-1}]$$

⌞ Hopf algebroid structure, and k unit, so has two multip

u some generator $(u) = \mathrm{Ext}^2_{k \overset{L}{\otimes}_A k}(k,k)$ (so $|u| = 2$)

$B \circlearrowright MF(X,\pi)$ ← (see Preygel's thesis)

u^{-1} kills perfect complexes

REnd acts on Coh of sing. fiber, so u^{-1} acts on MF.

B is aka E_2-Koszul dual of $A \to k$.

$k \overset{L}{\underset{A}{\otimes}} k$ NOT dg. alg. in simp comm. rings.

B has two diff't k-linear structures

- bracket 0
- def. of normal cone of A to ass gr

$\rightsquigarrow$ $B \to k[u,u^{-1}]$

Choice of u like choosing uniformizer - involves choices.

Rmk $\mathcal{B}$ E_2 $\Longrightarrow$ $\mathcal{B}$ ass mon. in non-comm sch

so makes sense to act on an object.

(**Thm 1**) $MF(X,\pi)$ is smth + proper over $\mathcal{B}$. (← new thing is the mixed char case.)

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l-adic realization:

I won't show how to Apply trace formula to compute $\chi(MF)$.
But will explain $\chi(MF) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}})$.

$\exists$ lax mon. functor

Kan extension

H: Set_ω "étale"?

$$\begin{tikzcd} \mathrm{Sch}^{nc}_A \arrow[r, "r"] & D^{const}_{ind}(\mathrm{Set}, \mathbb{Q}_\ell) \\ \mathrm{Sch}_A^{\text{can't read}} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

ind-constructible $\mathbb{Q}_\ell$-complexes.

$\exists$ via Deligne, etc?

lax monoidal $\Rightarrow$ extend to things over $\mathcal{B}$.

$$\mathrm{Sch}^{nc}_{\mathcal{B}} \longrightarrow r(\mathcal{B})\text{-modules in } D^{const}_{ind}(\mathrm{Set}, \mathbb{Q}_\ell)$$

Thm 2. $\chi_{r(\mathcal{B})}\big(r(MF(X,\pi))\big) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}})$

Inertia of Galois gp $G(K^{sp}/K)$

must act unipotently on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$.