Daniel Bergh: Categorical measures for equivariant derived categories

$/\mathbb{C}$, $G \circlearrowright X$, G finite, X variety, $D_G(\cancel{\text{coh}}\, X) = D^G(\text{coh}_G X)$,

and enhancements everywhere!

The (equivariant) Grothendieck group of varieties

$$K_0(Var) := \mathbb{Z}[Var/\cong] / \text{scissor}$$

↳ also has a presentation using smooth projective varieties and relations using blowups $\{Bl_Z X\} - \{E\} = \{X\} - \{Z\}$

$K_0(Var^G)$: same, things are invariant

Categorical measure

Δ has generators $[T]$, T saturated dg category

and relations using semiorthogonal decomposition

Tabuada: $\Delta \cong K_0^{(e)}(\mathbb{C})$

$K_0(Va) \xrightarrow{\mu} \Delta \qquad X \to D(X) \qquad$ ring map

$K_0(Var^G) \xrightarrow{\mu_G} \Delta \qquad X \to D_G(X) \qquad$ group map

Sometimes we only consider geometric dg categories for Δ

~~Is~~ $\Delta_{geo} \to \Delta$ ~~an~~ $\cong$?

There are zeta functions:

1) motivic zeta function on $K_0(Va)$

$$Z_{mot}(X, t) := \sum_{n \geq 0} [X^n/S_n] t^n$$

2) categorical zeta function on Δ, Δ_{geo}:

$$Z_{cat}(T, t) := \sum_{n \geq 0} [Sym^n T] t^n$$

for X smooth proj.: $\boxed{Z_{cat}(\mu(X), t) = \sum_{n \geq 0} \mu_{S_n}(X^n) t^n}$

⇝ λ-ring structure! but μ not a map of λ-rings...

Example $* = \mathrm{Spec}\, k$

$$Z_{mot}(*, t) = \frac{1}{1-t} \qquad Z_{cat}(M(?), t) = \prod_{k \geq 1} \frac{1}{1-t^k}$$

= representation theory of Sym_n!

The conjectures

1) "conjecture" 1 (Galkin-Shinder)

for any X: $\boxed{Z_{cat}(\mu(X), t) = \prod_{k\geq 1} \mu\left(Z_{mot}(X, t^k)\right)}$

OK up to dim 2

reduction to a "conjecture" in general

2) "conjecture" 2 (GS)

X G-variety, then $\boxed{\mu_G(X) = \sum_{g \in \mathrm{Conj}\, G} \mu\left(X^g / Z(g), t\right)}$

3) conjecture (Polishchuk - Van den Bergh)

$G \curvearrowright X$ effective, each $X^g/Z(g)$ smooth, then $D_G(X)$ has an SOD with terms $D(X^g/Z(g))$.

(stronger statement for a smaller class)

Theorem

1) true

2) false, but true enough for 1), at least if we pass to Δ_{geo}

3) false if effective is removed

The free case $G \curvearrowright X$

(2) $\Leftrightarrow$ $\mu_G(X) = \mu(X/G)$

using destackification

$$\begin{tikzcd} Y' \arrow[r, "\text{stacky blowups}"] \arrow[d] & \cdots \arrow[r] & Y \arrow[d] \\ Y_{cs} \arrow[r] & \cdots \arrow[r] & Y_{cs} \end{tikzcd}$$

Y smooth proper DM generically trivial

Y_{cs} smooth

$\Rightarrow$ SODs!

apply this to $Y = [X/G]$

<u>The non-effective case</u> $G \circlearrowright X$ has kernel N

$0 \to N \to G \to H \to 0$

s.t. H acts freely

conj (2) $\Leftrightarrow$ $\mu_G(X) = \mu_H(X \times \operatorname{Conj} N)$ diagonal action, H acts here

$= \mu(X \times \operatorname{Conj} N / H)$

<u>Proposition</u>: $\exists$ Azumaya algebra on $[X \times \overline{\operatorname{Irr}} N /H]$

s.t. $\operatorname{coh}_G X = \operatorname{coh} \cancel{X/A}\, A$, A at quotient

2 things can go wrong:

1) $\operatorname{Conj} N \neq \operatorname{Irr} N$ as H-reln

($\nexists$ no natural correspondence)

2) A not trivial always f.g.? / split

We are $\Delta_{geo} \xrightarrow{K_i^{top}} K_0^{\otimes}(Ab) \xrightarrow{\#()_{tors}} \mathbb{Q}^\times$

<u>Question</u>: Does K_i^{top} extend to Δ?

$\rightsquigarrow$ need f.g. for this?

Choose $G \circlearrowright X$ s.t. $[X/G]$ μ_n-gerbe over $X/G = Y$

<u>Theorem</u> (Bernardara) $P \to X/G$ corresponding bundle

$\mu(P) = \mu(X/G)$

<u>Ekedahl</u>: $\exists Y, \exists P$ s.t. $\{P\} \neq \{\mathbb{P}^{n-1} \times X\}$ in $K_0(\mathrm{Var})$

using torsion in cohomology!

$\to$ do this in K_i^{top}!