

Daniel Belyi: Categorical measures for equivariant derived categories

$k \subset G \curvearrowright X$, G finite, X variety, $D_G(\text{cat } X) = D^b(\text{cat}_G X)$
and enhancements everywhere!

The (equivariant) Grothendieck group of varieties

$$K_0(\text{Var}) = \mathbb{Z} \langle \text{Var}/\cong \rangle / \text{relations}$$

↳ also has a presentation using smooth projective varieties

and relations using blowups: $\{Bl_2 X\} - \{E\} = \{X\} - \{pt\}$

$K_0(\text{Var}^G)$: same things are invariant

Categorical measure

Δ has generators $[T]$ T natural in category

and relations using semiorthogonal decomposition

Takuroda: $\Delta \cong K_0^{\text{cat}}(\mathbb{C})$

$$K_0(\text{Var}) \xrightarrow{\text{cat}} \Delta \quad X \rightarrow D(X) \quad \text{ring map}$$

$$K_0(\text{Var}^G) \xrightarrow{\text{cat}} \Delta \quad X \rightarrow D_G(X) \quad \text{group map}$$

Sometimes we only consider geometric dg categories for Δ

$$\Delta_{\text{geo}} \rightarrow \Delta \quad \text{cat} \cong ?$$

There are zeta functions:

1) motivic zeta function on $K_0(\text{Var})$

$$Z_{\text{mot}}(X, t) := \sum_{n \geq 0} [X^n / \text{Sym}^n X] t^n$$

2) categorical zeta function on Δ , Δ_{geo}

$$Z_{\text{cat}}(T, t) := \sum_{n \geq 0} [\text{Sym}^n T] t^n$$

for X smooth proj:

$$Z_{\text{cat}}(\mu(X), t) = \sum_{n \geq 0} K_{\text{Sym}^n}^{\text{cat}}(X^n) t^n$$

→ \mathbb{Z} -ring structure, but μ not a map of \mathbb{Z} -rings...

Example $s = \text{Spec } k$

$$Z_{\text{mot}}(X, t) = \frac{1}{1-t}$$

$$Z_{\text{cat}}(X, t) = \prod_{k \geq 1} \frac{1}{1-t^k}$$

= representative theory of Sym!

The conjectures

1) "conjecture" (Gallier-Skinner)

$$\text{for any } X: \left| Z_{\text{cat}}(X, t) = \prod_{k \geq 1} P(Z_{\text{mot}}(X, t^k)) \right|$$

OK up to date?

reduction to a "conjecture" in general

2) "conjecture" (GS)

$$X \text{ } G\text{-variety, then } \left| P_G(X) = \sum_{g \in \text{Con } G} P(X^g / Z(g), t) \right|$$

3) conjecture (Polishchuk - Van de Beryk)

$G \curvearrowright X$ effective, each $X^g / Z(g)$ smooth, then $\underline{P}_G(X)$

has an SOD with terms $D(X^g / Z(g))$

(stronger statement for a smaller class)

Theorem

1) true

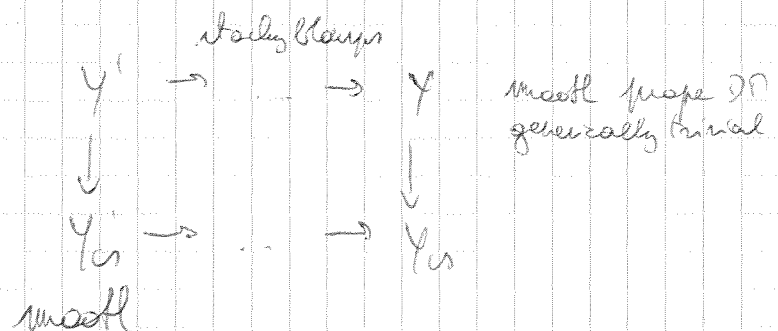
2) false, but true enough for 1), at least if we pass to Δ_{geo}

3) false in effectiveness removed

The free case $G \curvearrowright X$

$$\text{2) } \Leftrightarrow P_G(X) = P(X/G)$$

any detorsion



⇒ SODs!

apply this to $Y = [X/G]$

The non-effective case $G \curvearrowright X$ has kernel N

$$0 \rightarrow N \rightarrow G \rightarrow H \rightarrow 0$$

$\Rightarrow H$ acts freely

$$\text{conj } (2) \Leftrightarrow \begin{aligned} \mathbb{P}_G(X) &= \mathbb{P}_H(X \times_{\text{diag action}} \text{conj } N) \\ &= \mathbb{P}(X \times \text{conj } N / H) \end{aligned}$$

Proposition: \exists Azumaya algebra on $[X \times_{\text{conj } N} N/H]$

$$\cong \text{col}_G X \cong \text{col } X/A \text{, } A \text{ is quotient}$$

2 things can go wrong:

1) $\text{conj } N \neq \text{im } N$ as H -rebr

(\exists no natural correspondence)

2) \neq not trivial, always f.g. / y/lit

We use $\mathbb{P}_G(X) \xrightarrow{U_i^{\text{top}}} \mathbb{P}_0(A_G) \xrightarrow{\neq 0} \mathbb{Q}^{\times}$

Question: Does U_i^{top} extend to Δ ?

is need f.g. for this?

choose $G \curvearrowright X$ & $[X/G]$ \mathbb{P}_n -gerbe are $X/G = Y$

Theorem (Bourbaki) $P \rightarrow X/G$ corresponding bundle

$$p(P) = p(X/G)$$

Fredholm: $\exists Y, \exists P \neq \mathbb{P}^{n-1} \times X/G \in U_0(\text{Var})$

why torsion in cohomology!

\rightarrow do this in U_i^{top} !