AN ELEMENTARY LG MODEL WITHOUT PROJECTIVE MIRRORS

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A	B
symplectic	complex alg.

Goal: Construct symplectic Lefschetz fibrations

Lie theory:

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G semisimple Lie group. $\mathfrak{g}$, $\mathcal{H}$ = hermitian form on $\mathfrak{g}$, $\Omega = \operatorname{im} \mathcal{H}$

Adjoint orbit of $H_0 \in \mathfrak{h}$: $\mathcal{O}(H_0) = \{ g H_0 g^{-1}, g \in G \}$

H regular

<u>Thm</u>: (–, Grama, San Martín)
The height function f_H gives $\mathcal{O}(H_0)$ the structure of a symplectic Lefschetz fibration.

$$f_H : \mathcal{O}(H_0) \longrightarrow \mathbb{C}$$
$$A \longmapsto \langle A, H \rangle$$

Weyl group

The critical points are $W \cdot H_0$

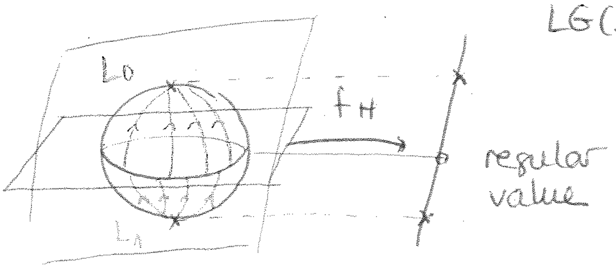
<u>Prop</u>: $\mathcal{O}(H_0)$ has the diffeomorphism type of $T^*\mathrm{Flag}$

Example: $H_0 = \begin{pmatrix} n & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$ $\qquad \mathcal{O}(H_0) \underset{\text{dif}}{\sim} T^*\mathbb{P}^n$

Example:
$SL(2)$, $\quad H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$X = \mathcal{O}(H_0) \underset{\text{dif}}{\sim} T^*\mathbb{P}^1$$

LG(2)



Prop: The Fukaya-Seidel category Fuk LG(2) is generated by 2 lagrangians L_0, L_1

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i<j \\ \mathbb{Z} & i=j \\ 0 & i>j \end{cases}$$

$m_k = 0$ except for m_2.

We look for an algebraic variety Y such that $D^b(\mathrm{Coh}\, Y) \simeq$
$\simeq \mathrm{Fuk}\, LG(2)$

- Prop: $LG(2)$ has no projective mirrors.
- Prop: X compactifies symplectically and holomorphically to $\mathbb{P}^1 \times \mathbb{P}^1$?

Complex structure of X, symplectic structure of X

$Z_2 : \mathrm{Tot}(T^*\mathbb{P}^1) = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$ with canonical structures

$H^1(Z_2, TZ_2) = \mathbb{C} \ni \sigma \neq 0$

$\underset{\text{biholom.}}{X \sim} \mathcal{Z}_2(\sigma)$ $\qquad$ S^2 is a Lagrangian submanifold of X

Deformations of $Z_k = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$

$H^1(Z_k, TZ_k) = \mathbb{C}^{k-1}$

Tool: deform vector bundles on Z_k

Lemma: Every holom. bundle on Z_k is filtrable and algebraic.

Corollary: Rank 2 bundles on Z_k are extensions of line bundles.

For TZ_k

$Z_k = U \cup V$ $\qquad$ in $\cap$

$(z,u) \quad (\xi, v) = (z^{-1}, z^k u)$

TZ_k has transition matrix $\begin{pmatrix} z^k & kz^{k-1}u + z^k\sigma \\ 0 & -z^{-2} \end{pmatrix}$

$$0 \to \mathcal{O}(-k) \to TZ_k \to \mathcal{O}(2) \to 0$$

$\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}(-k)) = \mathbb{C}^{k+1}$

$\cong$

$H^1(Z_k, \mathcal{O}(-k+2))$ generated by $\sigma \in \underset{\text{trivial}}{z^{-1}u}, \underbrace{\cancel{z^{-k-1}}, \ldots, z^{-1}}_{\text{give all the deformations}}$ this cannot

- Prop: The moduli space of rank 2 bundles on Z_k with local $c_2 = j$ is a quasi-projective variety of dimension $2j-k-2$.
  There are embeddings $\mathfrak{M}_j(Z_k) \longrightarrow \mathfrak{M}_{j+k}(Z_k)$

- Prop: Let $Z_k(\sigma)$ be a nontrivial deformation of Z_k. Then every holomorphic vector bundle on $Z_k(\sigma)$ splits.

- Prop: $Z_k(\sigma)$ is affine
  (Barmeier)

$X \subset \mathfrak{sl}(2) = \mathbb{C}^3$

$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in X$ iff it has eigenvalues ± 1

$X: \quad x^2 + yz - 1 = 0 \quad$ in $\mathbb{C}^3$

We extend the potential f_H to $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ as a rational map it extends to $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{R_H} \mathbb{P}^1$

$$[x:y],[z:w] \longmapsto \left[\frac{xz+yw}{xz-yw} : 1\right]$$

Standard blow-up construction then gives a holomorphic map

$F_H : \overline{\Gamma} \longrightarrow \mathbb{P}^1$, $\overline{\Gamma}$ compactification $\qquad \overline{LG(2)} = (\overline{\Gamma}, F_H)$

Prop: The critical points of F_H are the same as those of f_H

Corollary: $\overline{LG(2)}$ has no projective mirrors.

<u>3-folds</u>: $W_k = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2))$

W_1 has no commutative defs

$W_2 = \mathrm{Tot}(\mathcal{O}(-2) \oplus \mathcal{O})$ has 1-dim def. space

$H^1(W_i, TW_i)$ is infinite dimensional $i \geq 3$

For $i \geq 4$ deformations are obstructed