Kaledin || How to glue derived categories

Context: 1) $\mathcal{C}_i$ abelian, $\varphi: \mathcal{C}_1 \to \mathcal{C}_2$ left exact.

$$\mathcal{C}_1 \amalg_\varphi \mathcal{C}_2 := \text{cat. w/ ob } (C_1, C_2, \alpha),\ \alpha: C_2 \to \varphi(C_1)$$

aka comma category; abelian again.

Then $\mathcal{D}(\mathcal{C}_1 \amalg_\varphi \mathcal{C}_2) \simeq \langle \mathcal{D}(\mathcal{C}_1), \mathcal{D}(\mathcal{C}_2) \rangle$ w/ gluing functor $R^\cdot\varphi$.

Issue: cannot recover $\mathcal{D}(\mathcal{C}_1 \amalg_\varphi \mathcal{C}_2)$ from $\mathcal{D}(\mathcal{C}_i)$, $R^\cdot\varphi$.

To find enhancement, usually need some context.

2) I small, $\mathcal{C}$ ab. $\mathcal{C}^I$ ab. But $\mathcal{D}(\mathcal{C}^I)$ cannot be recovered from $\mathcal{D}(\mathcal{C})$, I. (I an arrow $\Rightarrow$ last example)

3) Combine 1)+2): I small, $\mathcal{C}_i$ abelian, $f: i \to i' \mapsto f^*: \mathcal{C}_{i'} \to \mathcal{C}_i$ left ex

$$f_1^* \circ f_2^* \to (f_2 \circ f_1)^*$$

ie, Groth. constrn $\begin{array}{c}\mathcal{C}\\ \downarrow\\ I\end{array}$ "prefibration" conditions.

Consider category of sections $\sigma: I \to \mathcal{C}$, $Sec(I, \mathcal{C})$

Note $\mathcal{C}$ not abelian. Then $Sec(I, \mathcal{C})$ is abelian.

But cannot construct $\mathcal{D}(Sec(I, \mathcal{C}))$ from I, $\mathcal{D}(\mathcal{C}_i)$.

Typically, 4 enhancements:

1) DG enhancement, via model cat of Tabuada + Morita equiv. (k-linear, k base only)

2) Spectral enhancement

3) Stable model cat.

4) Stable ∞ cat.

Each gives soln to problem. But are they applicable?

People use 1.)$^{\text{dg}}$ in alg geom. But 1.) doesn't apply well to ex 2; stable mps, etc. ex 3 even worse.

Even worse for 2.)$^{\text{spectral}}$.

3) has advantage: just a category. But not flexible enough; not all of Δ'd cats in nature come from stable model cat in practice, in an obvious way.

4) designed for this, but still seems hard. Are there easier ideas in practice?

Today: something btw 3 and 4. Relax notion of "model," not of "stable"

<u>Defn</u> A model pair is $(\mathcal{C}, \mathcal{C}')$ st

- $\mathcal{C}'$ model cat (finite (co)lims, $\overset{\text{cofib}}{C}, \overset{\text{fib}}{F}, \overset{\text{weak}}{W}$)
- $\mathcal{C} \subset \mathcal{C}'$ full closed under W ($W: \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$, $c \in \mathcal{C} \Leftrightarrow c' \in \mathcal{C}$)

Then $Ho(\mathcal{C}) \underset{\text{full}}{\subset} Ho(\mathcal{C}')$; no need to demand $\mathcal{C}$ is model cat itself. (In particular, may not have all (co)lims)

Def $(\mathcal{C}, \mathcal{C}')$ is stable if

0. pointed — $\exists$ zero object, $0 \in \mathcal{C} \subset \mathcal{C}'$

1. Given
$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ X' & \to & Y' \end{array}$$
w/ $X, Y \in \mathcal{C}$, htpy Cartesian.
Then it's htpy co Cartesian.

(Likewise, dual version: works in op category.
So co Cartes $\Rightarrow$ Cartes.)

2. For such a square,
$$X, Y, X' \in \mathcal{C} \Rightarrow Y' \in \mathcal{C}$$
$$X', Y', Y \in \mathcal{C} \Rightarrow X \in \mathcal{C}.$$

$$\begin{array}{ccc} c & \to & c \\ \downarrow & & \\ c & & \end{array} \Rightarrow \text{po in } C$$

$$\begin{array}{ccc} & & c \\ & & \downarrow \\ c & \to & c \end{array} \Rightarrow \text{pb in } C.$$

Prop $(\mathcal{C}, \mathcal{C}')$ stable $\Rightarrow$ $\mathrm{Ho}(\mathcal{C})$ triangulated

But $\mathrm{Ho}(\mathcal{C}')$ need not be triangulated!

Def $\varphi : \mathcal{C}_1' \to \mathcal{C}_2'$ is right durable if $\varphi(F) \subset F$, $\varphi(F \cap W) \subset \varphi(F \cap W)$

φ is stable if $R\varphi$ preserves squares as in 1, and $\varphi(\mathcal{C}_1) \subset \mathcal{C}_2$.

Prop $(\mathcal{C}_i, \mathcal{C}_i')$ stable model pair, φ stable right durable.

Then $(\mathcal{C}_1 \amalg_\varphi \mathcal{C}_2, \mathcal{C}_1' \amalg_\varphi \mathcal{C}_2')$ is stable model pair.

Pf (0) fin (co)lims. Just a fact about model cats: Given I finite, functors $I \xrightarrow{c_i} C_i$, $c_2 \xrightarrow{\alpha} \varphi(c_1)$,

then $\operatorname{colim}_I c_2 \underset{\text{maps}}{\xrightarrow{\alpha}} \operatorname{colim}_I \varphi(c_i) \to \varphi(\operatorname{colim}_I c_i)$, is the colim.

But not so easy for limits: $\lim_I c_2 \to \lim_I \varphi(c_I) \leftarrow \varphi(\lim_I c_1)$ $\rightsquigarrow^{\text{fiber product}}$ $(\lim_I c)_2$ is the $\underset{\text{2nd}}{\text{the}}$ component of $\lim_I C$.

component 1 of $\lim_I$.

So lim no longer pointwise.

lim = fiber product $\to \varphi(\lim_I c_1)$.

And so on. //

④

Ex 2 has problem $\mathcal{C}^I$ has no model structure in general, but should impose condition on $\mathcal{C}$, OR on I.
Imposing condition on I is Reedy condition.

Defn A Reedy category I is
- small cat, w/
- degree fcn $\deg : I \longrightarrow \mathbb{N}$
- two classes of maps L, M s.t. $L \subset I$, $M \subset I$ subcategories,

(1) $L \ni i' \to i$ $\deg(i') \leq i$
$M \ni i' \to i$ $\deg(i') \geq i$

(2) Any f uniquely decomposes as ~~lower, then raise~~:
$f = l \circ m$ $l \in L$, $m \in M$

Note • I Reedy $\Rightarrow$ I^{op} Reedy

• Obj have no autom,
• only one obj in each isom class (no isom).

Thm (Reedy) I Reedy, $\mathcal{C}$ model $\Rightarrow$ $\mathcal{C}^I$ has natural model str.

Ex • $I = \Delta$
• $I = \Delta^{op}$.

Defn A Reedy model prefibration $\mathcal{C} \to I$ is
- pullback prefib
- I Reedy
- Fibers $\mathcal{C}_i$ have model str.
- Transition functors $f^*: \mathcal{C}_i \to \mathcal{C}_{i'}$ are right derivable
- l^* has right adj $\forall\, l \in L$
- $(l^* \circ m^*) \to (m \circ l)^*$ is an isom.

No adjunction condition for m^*, but these are symmetric.

<u>Thm</u> (Balzin) $\mathcal{C} \to I$ Reedy model prefibr.

Then $Sec(I, \mathcal{C})$ has natural model structure.

<u>Applications</u>:

- I small. ~~[illegible]~~ nerve, aka "simp replacement":

$$\Delta I = \text{cat w/ ob } ([n] \in \Delta,\ [n] \xrightarrow{\varphi} I)$$

$$\begin{tikzcd} & \Delta I \arrow[dl, "p"', "\text{last elt}"] \arrow[d, "\text{forget}"] \\ I & \Delta \end{tikzcd}$$

$$\begin{tikzcd} & (i_0, \dots, i_n) \arrow[dl, mapsto, "p"'] & \\ i_n & & I_{[n]}. \end{tikzcd}$$

[Rmk Reedy str more flexible than "cof. finitely generated" str, which is trickier today]

ΔI is Reedy, pulled back from Δ.

So take

$$\begin{tikzcd} & \mathcal{C} \arrow[d] \\ \Delta I \arrow[r, "p"'] \arrow[ur, dashed] & I \end{tikzcd}$$

$Sec(\Delta I, \mathcal{C})$ better than $Sec(I, \mathcal{C})$. Can call this "$RSect(I, \mathcal{C})$"

where $RSec(I, \mathcal{C}) \subset Sec(\Delta I, \mathcal{C})$.

Thm $RSec(I, \mathcal{C}) \subset Sec(\Delta I, \mathcal{C})$ is stable model pair!

- Can also apply to (co)monad modules.
- Any dg cat gives $(\mathcal{C}, \mathcal{C}')$ stable model pair, and can also return to dg world.