

Kaledin

How to give derived categories

①

Context 1)  $\mathcal{C}_i$  abelian,  $\varphi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  left exact.

$$\mathcal{C}_1 \amalg_{\varphi} \mathcal{C}_2 := \text{cat. w/ ob } (\mathcal{C}_1, \mathcal{C}_2, \alpha), \alpha: \mathcal{C}_2 \rightarrow \varphi(\mathcal{C}_1)$$

also commut category; abelian again.

Then  $\mathcal{D}(\mathcal{C}_1 \amalg_{\varphi} \mathcal{C}_2) \simeq \langle \mathcal{D}(\mathcal{C}_1), \mathcal{D}(\mathcal{C}_2) \rangle$  w/ gluing functor  $R^*\varphi$ .

Issue: cannot recover  $\mathcal{D}(\mathcal{C}_1 \amalg_{\varphi} \mathcal{C}_2)$  from  $\mathcal{D}(\mathcal{C}_1), R^*\varphi$ .

To find enhancement, usually need some context.

2)  $\mathcal{I}$  small,  $\mathcal{C}$  ab.  $\mathcal{C}^{\mathcal{I}}$  ab. But  $\mathcal{D}(\mathcal{C}^{\mathcal{I}})$  cannot be recovered from  $\mathcal{D}(\mathcal{C}), \mathcal{I}$ . ( $\mathcal{I}$  an arrow  $\Rightarrow$  last example)

3.) Combine 1+2):  $\mathcal{I}$  small,  $\mathcal{C}_i$  abelian,  $f: i \rightarrow i' \mapsto f^*: \mathcal{C}_{i'} \rightarrow \mathcal{C}_i$  left ex

ie, Groth. condition  $\begin{matrix} \mathcal{C} \\ \downarrow \\ \mathcal{I} \end{matrix}$  "preservation" conditions.  $f_1^* \circ f_2^* \rightarrow (f_2 \circ f_1)^*$

Consider category of sections  $\Gamma: \mathcal{I} \rightarrow \mathcal{C}, \text{Sec}(\mathcal{I}, \mathcal{C})$

Note  $\mathcal{C}$  not abelian. Then  $\text{Sec}(\mathcal{I}, \mathcal{C})$  is abelian.

But cannot construct  $\mathcal{D}(\text{Sec}(\mathcal{I}, \mathcal{C}))$  from  $\mathcal{I}, \mathcal{D}(\mathcal{C}_i)$ .

Typically, 4 enhance 3:

- 1) DG enhancement, via model cat of Tabuada + Munkers eqns  
(k-linear, k base ring)
- 2) Special enhancement
- 3) Stable model cat.
- 4) Stable cocat.

Each gives soln to problem. But are they applicable?

People use 1) <sup>dg</sup> in alg geom. But 1) doesn't apply well to ex 2; shuffle ops, etc.  
ex 3 even worse.

Even worse for 2) <sup>special</sup>.

3) has advantage: just a category. But not flexible engh; not all of  $\Delta^d$  cats in nature have hom stable model cat in practice, in an obvious way.

4) designed for this, but still seems hard. Are there easier ideas in practice?

Today: something btw 3 and 4. Relax notion of "model," not of "stable."

Def A model pair is  $(\mathcal{C}, \mathcal{C}')$  st

- $\mathcal{C}'$  model cat (finite (co)lims,  $\mathcal{C}, F, W$ )
- $\mathcal{C} \subset \mathcal{C}'$  full closed under  $W$  ( $W: \mathcal{C} \rightarrow \mathcal{C}'$ ,  $C \in \mathcal{C} \Leftrightarrow C' \in \mathcal{C}$ )

Then  $\text{Ho}(\mathcal{C}) \subset_{\text{all}} \text{Ho}(\mathcal{C}')$ ; no need to demand  $\mathcal{C}$  is model cat itself. (In practice, may not have all (co)lims)

Def  $(\mathcal{C}, \mathcal{C}')$  is stable if

0. pointed  $\rightarrow \exists$  zero object,  $0 \in \mathcal{C} \subset \mathcal{C}'$

1. Given 
$$\begin{array}{ccc} X & \rightarrow & Y \\ \downarrow & & \downarrow \\ X' & \rightarrow & Y' \end{array}$$
 w/  $X, Y \in \mathcal{C}$ ,  $\mathcal{C}$  <sup>htpy</sup> Cartesian.  
Then it's htpy co Cartesian.

(Likewise, dual version: works in op category.  
So coCart  $\rightarrow$  Cart.)

2. For such a square,

$$\begin{aligned} X, Y, X' \in \mathcal{C} &\Rightarrow Y' \in \mathcal{C}' \\ X', Y', Y \in \mathcal{C}' &\Rightarrow X \in \mathcal{C}. \end{aligned}$$

$$\begin{array}{ccc} \mathcal{C} & \rightarrow & \mathcal{C} \\ \downarrow & & \rightarrow \text{po in } \mathcal{C} \\ \mathcal{C}' & & \end{array}$$

$$\begin{array}{ccc} \mathcal{C}' & \rightarrow & \mathcal{C}' \\ \downarrow & & \rightarrow \text{pb in } \mathcal{C}. \\ \mathcal{C} & \rightarrow & \mathcal{C} \end{array}$$

Prop  $(\mathcal{C}, \mathcal{C}')$  stable  $\Rightarrow \text{Ho}(\mathcal{C})$  triangulated

But  $\text{Ho}(\mathcal{C}')$  need not be triangulated!

Def  $\varphi: \mathcal{C}_1' \rightarrow \mathcal{C}_2'$  is right dual of  $\varphi(F) \subset F$ ,  $\varphi(Fw) \subset \varphi(Fw)$

$\varphi$  is stable if  $R\varphi$  preserves squares as in 1, and  $\varphi(\mathcal{C}_1) \subset \mathcal{C}_2$ .

Prop  $(\mathcal{C}_1, \mathcal{C}_1')$  stable model pair,  $\varphi$  stable right dual.

Then  $(\mathcal{C}_1 \cup_{\varphi} \mathcal{C}_2, \mathcal{C}_1' \cup_{\varphi} \mathcal{C}_2')$  is stable model pair.

PF (C) in (colims). Just a fact about model cats: Given  $I$  finite, arrows  $I \xrightarrow{G_i} \mathcal{C}_i$ ,  $\mathcal{C}_2 \xrightarrow{\alpha} \varphi(\mathcal{C}_1)$ ,

then  $\text{colim}_I \mathcal{C}_2 \xrightarrow[\text{maps}]{\alpha} \text{colim}_I \varphi(\mathcal{C}_i) \rightarrow \varphi(\text{colim}_I \mathcal{C}_i)$ , is the colim.

But not so easy for limits:

$$\begin{array}{ccc} \lim_I \mathcal{C}_2 & \xrightarrow{\alpha} & \lim_I \varphi(\mathcal{C}_i) \xrightarrow{\varphi(\lim_I \mathcal{C}_i)} \varphi(\lim_I \mathcal{C}_i) \\ & \searrow & \downarrow \text{compact } I \\ & & \lim_I \varphi(\mathcal{C}_i) \end{array}$$

the product  $(\lim_I \mathcal{C})_2$  is the compact of  $\lim_I \mathcal{C}$ .

So  $\lim$  no longer pointwise.

$$\lim = \text{the product} \rightarrow \varphi(\lim_I \mathcal{C}_i).$$

And so on //

Ex 2 has problem:  $\mathcal{C}^I$  has no model structure in gen, but should impose conditions on  $\mathcal{C}$ , or on  $I$ .  
Imposing condition on  $I$  is Reedy condition.

Def A Reedy category  $I$  is

- small cat,  $\omega_1$
- degree fun  $\text{deg}: I \rightarrow \mathbb{N}$
- two classes of mps  $L, M$  s.t.  $L \subset I, M \subset I$  subcategories,

$$(1) \begin{aligned} L \ni i' \rightarrow i & \quad \text{deg}(i') \leq i \\ M \ni i' \rightarrow i & \quad \text{deg}(i') \geq i \end{aligned}$$

(2) Any  $f$  uniquely decomposes as ~~two~~ two  $f = l \circ m$   $l \in L, m \in M$ .

$f = l \circ m \quad l \in L, m \in M$

Note:  $I$  Reedy  $\Rightarrow I^{op}$  Reedy

- Obj have no autom,
- only one obj in each isom class (no isom).

Thm (Reedy)  $I$  Reedy,  $\mathcal{C}$  model  $\Rightarrow \mathcal{C}^I$  has natural model str.

Ex  $I = \Delta$

$I = \Delta^{op}$ .

Def A Reedy model prebitar  $\mathcal{C} \rightarrow I$  is

- Grothendieck pre fib
- $I$  Reedy
- Fibers  $\mathcal{C}_i$  have model str.
- Transition functors  $f^*: \mathcal{C}_i \rightarrow \mathcal{C}_j$  are right derivable
- $L^*$  has right adj  $\forall l \in L$
- $(L^* \circ_{M^*}) \rightarrow (mod L)^*$  is an isom.

No adjunctions either for  $M^*$ , but these are symmetric.

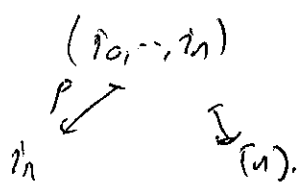
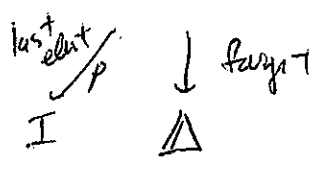
Thm (Balzin)  $\mathcal{C} \rightarrow \mathcal{I}$  Reedy model presheaves

Then  $\text{Sec}(\mathcal{I}, \mathcal{C})$  has natural model structure.

Applications:

-  $\mathcal{I}$  small. ~~Reedy~~ nerve, aka "simp replacement"

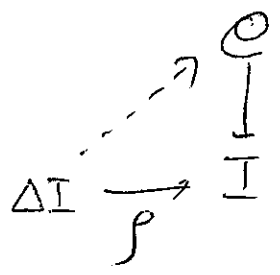
$$\Delta \mathcal{I} \rightleftharpoons \text{cat w/ ob } \{[n] \in \Delta, [n] \xrightarrow{p} \mathcal{I}\}$$



Frank  
 Reedy sets more flexible than "arbitrarily generated" sets, which is tricker today

$\Delta \mathcal{I}$  is Reedy, pulled back from  $\Delta$ .

So take



$\text{Sec}(\Delta \mathcal{I}, \mathcal{C})$  better than  $\text{Sec}(\mathcal{I}, \mathcal{C})$ . Can all the "Rect( $\mathcal{I}, \mathcal{C}$ )" where  $\text{Rect}(\mathcal{I}, \mathcal{C}) \subset \text{Sec}(\Delta \mathcal{I}, \mathcal{C})$ .

Then  $\text{Rect}(\mathcal{I}, \mathcal{C}) \subset \text{Sec}(\Delta \mathcal{I}, \mathcal{C})$  is stable model pair!

- Can also apply to (co)monad models.

- Any by cat gives  $(\mathcal{C}, \mathcal{C}')$  stable model pair, and can also return to dg world.