# Kapranov | Perverse schobers on surfaces via Ran categories

Perv schobers = categorical analogue of perverse sheaves,

$$\begin{tikzcd} \text{Vect} & \text{Pre}\Delta\text{'d cats} \arrow[l, leftsquigarrow] \arrow[l, bend left, "K_0\otimes\mathbb{Q}"] \end{tikzcd}$$

Usually, perv sheaf is a <u>complex</u> of sheaves — not obvious how to generalize.

So use description of cat of perv shvs.

<u>Motivation</u>: Coefficients for forming Fukaya categories

↗ Schobers are

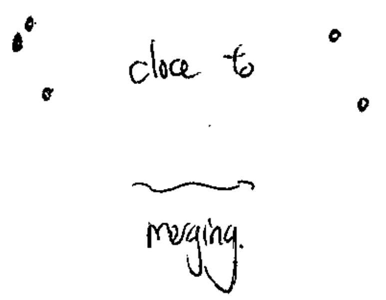
## ① Ran spaces + Ran categories

bread + butter of fact. alg.

X top space, locally compact.

$\text{Ran}(X) := \{\text{finite, non empty } A \subset X\}$. If X metric space, topologize by Hausdorff metric:

$$\rho_{\text{Haus}}(A,B) = \max_a \min_b \rho(a,b).$$



Filtration by cardinality

$\text{Ran}^{=d} = \text{Sym}^d_{\neq}(X)$.

$$\text{Ran}^{\leq 1} \subset \text{Ran}^{\leq 2} \subset \cdots$$

$\mathrm{Ran}^{=d}$ is mfld if $X$ is.

Study sheaves on $\mathrm{Ran}(X)$ constructible wrt $\mathrm{Ran}^{\le d}$.

Ex $X = \mathbb{R}$, $\mathrm{Ran}^{=d} = \{t_1 < \cdots < t_d\}$
a cell.

$\mathcal{F}$ constr. $\Longrightarrow$ data $(F_d)_{d \geq 1}$, $\sigma_i : F_d \longrightarrow F_{d+1}$

$$F_1 \to F_2 \rightrightarrows F_3 \mathrel{\substack{\to\\ \to\\ \to}} F_4$$

no faces, only degeneracies. faceless cosimplicial object (surjections = merging)

MacPherson $\mathcal{Y} = (Y, (Y_\alpha))$ stratified. Then

$\mathrm{Ex}(\mathcal{Y})$ = exit path cat; combines fund. gpd $\Pi_1(\mathcal{Y}_\alpha)$
so that

$$\mathrm{Constr}(\mathcal{Y}) \simeq \mathrm{Fun}(\mathrm{Ex}(\mathcal{Y}), \mathrm{Vect})$$

$$\mathrm{Ex}(\mathrm{Ran}(\mathbb{R})) \simeq (\Delta_{\mathrm{surj}})^{op}$$

(A: not "unital" in case of alg having unit)

$\downarrow$

"Unital" version (Lurie, Gaitsgory)

extending space is hard; extend the category.

$X$ mfld. $R^0(X) :=$ poset w/ obj $\coprod$ open balls $\subset X$
hom inclusions.

③

Ran category: $\mathcal{R}(X) = \mathcal{R}^0(X)\,[W^{-1}]$

$\uparrow$

localize wrt inclusions
which are htpy equiv,
inducing bijection on $\pi_0$ of balls

$\mathcal{R}(X)$ contains $Ex(\mathrm{Ran}(X))$
(surjective on $\pi_0$ inclusions)

**Ex** 1) $\mathcal{R}(\mathbb{R}) = \Delta^{op}$

(a cyclic extension of
cyclic cat $\Lambda$)

2) $\mathcal{R}(S^1) = (\Lambda^{\mathbb{R}})^{op} \cong \Lambda^{\mathbb{R}}$ ← paracyclic cat of Connes

A paracyclic obj is
$$\begin{tikzcd} F_{n-1} \arrow[r, shift right, "s_i"'] & F_n \arrow[l, shift right, "\partial_i"'] \arrow[loop below, "\tau_n"] \end{tikzcd}$$
$i=0,\dots,n$

$\tau_n$ rotates $\partial_i, s_i$.
$(\tau_n)^{n+1}$ central elmt.

$\tau_n^{n+1} = id$ is a cyclic object?

② Perverse sheaves on a disk, and paracyclic/Ran data.

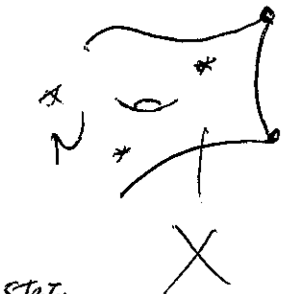
$\mathcal{Y}$ strat $\mathbb{C}$ mfld, $\underset{ab}{\mathrm{Perv}(\mathcal{Y})} \subset \mathcal{D}^b_{const}(\mathcal{Y})$

$\uparrow$ const cohom.

Case: $\mathcal{Y} = (\underset{\text{Riem surf}}{X}, \underset{\text{finite set}}{N})$. $X$ can have $\partial$, corners.

$\uparrow$ oriented, topological

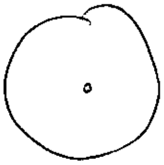
$N$ on interior



$\partial$, corners NOT
low-dim strata, not ~~open part~~ part of generic stratum.

Classical desc. of

$\mathrm{Perv}(\mathbb{D}, 0)$



$|z| < 1$.

$\Vert$

diagrams
$$\begin{tikzcd} \Phi \arrow[r, shift left, "a"] & \Psi \arrow[l, shift left, "b"] \end{tikzcd}$$

nearby cycles        vanishing cycles,

s.t. $T_\psi = 1 - ab$, $T_\varphi := 1 - ba$ are invert.

$\rightsquigarrow$

paracyclic interp.   $[\Psi \xrightarrow{b} \Phi]$ ← consider as gpd

$\Vert$

$[b]$ as gpd obj are elts of $\Phi$

$\hom(\varphi, \varphi')$

$\Vert$

$\{\psi \mid b(\psi) = \varphi' - \varphi\}$.

N. $[b]$ (via D-K) is simp obj in Vect.

$$\begin{tikzcd} \varphi_0 \arrow[r, bend left, "b"] & \varphi_1 \arrow[r, bend left, "b"] & \varphi_2 \end{tikzcd}$$

**Prop** (A) Given any linear map $b : \Psi \to \Phi$,
have bijection btw

(i) extensions of $b$ to a diag representing $\mathcal{F} \in \mathrm{Perv}(D,0)$

(ii) extending $\underbrace{\text{simplicial str}}_{\text{on } N_\bullet[b]}$ to paracyclic str

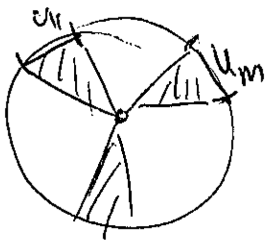
(B) i.e.,

$\mathrm{Perv}(D,0) \simeq \{$paracyclic vec. spaces which, as simp vec spaces, are $N_\bullet[b]\}$ (ie, are 1-Segal)

Nothing is derived here.

**Pf sketch** of (B) Given $\mathcal{F} \in \mathrm{Perv}(D,0)$, make functor on $\mathrm{Ran}(S^1) = \partial D$
$\|$
circle of dirs @ 0.

Given $U = \coprod U_i$, disj union of arcs on $S^1$,



Cone $= K(U)$

purity (!)

Thus $\boxed{R\Gamma_{K(U)}(D, \mathcal{F}) \text{ concent. in one degree, degree 1.}}$

purity property of perverse sheaves

So cond $U \mapsto \mathbb{H}^1_{K(U)}(U, \mathcal{F})$.

Purity property $\Longrightarrow$ more suitable for Fuk. cat.

Rmk

$K =$  is okay, but not  .

③ For a surface: Relative Ran cat.

$\mathrm{Perv}(X,N) \ni \mathcal{F} \rightsquigarrow$ loc. sys. on $X \setminus N$ and $(\Phi,\Psi)$ data @ each pt.

$||$

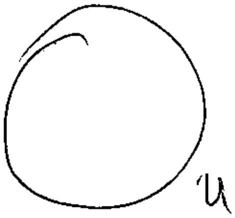
$\mathrm{Funct}(\ R(X,N), \mathrm{Vect}\ )$

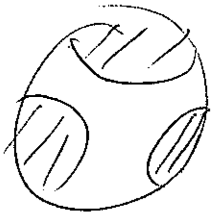
relative Ran cat.

Start w/ $R^0(X,N)$: poset of pairs $U' \subset U$

$U'$ disj balls

$U$ ball in $X$

s.t. 



$U'$ shaded

$K = U \setminus U'$ contractible, and $|K \cap N| \leq 1$

(!) CONT'D: For $\mathcal{F} \in \mathrm{Perv}(X,N)$

$$\mathbb{H}^{\neq 1}_{K}(U, \mathcal{F}) = 0$$

$$R(X,N) := R^0(X,N)[W^{-1}]$$

└ inclusions $(U' \subset U) \subset (V' \subset V)$
st
- $U' \hookrightarrow V'$ h. eq.
- $(U \setminus U') \cap N = (V \setminus V') \cap N$.

<u>Thm</u> Have ~~equiv~~

$$\mathrm{Perv}(X,N) \xhookrightarrow[\text{fully faith}]{} \mathrm{Fun}(R(X,N), \mathrm{Vect})$$

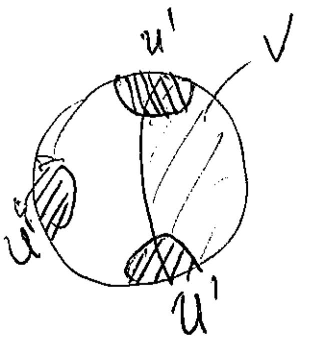
$$\rotatebox{90}{$\in$} \qquad F$$

where $F$ satisfies
(i) $F$ sheaf wrt coverings: $U = \bigcup U_i$, $U_i' = U_i \cap U'$

(ii) <u>exactness</u> (2-Segal) — (gluing of $\mathrm{Perv}(D,0)$)
$\forall\ (U' \subset U) \in R$ and $V \subset U$ disk

s.t. $(V \cup U', U)$, $(V \cap U', U)$ $\in R$

then
$F(V \cup U', U) \longrightarrow F(U', U) \longrightarrow F(V \cap U', U)$
is SES.



cf. Fary functor's description (MacPherson)

Purely abelian descrip, can categorify:

$$\begin{tikzcd} \mathrm{Perv}(\mathcal{D},0) \arrow[r, squiggly] & \text{Sph functors} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{D}_0 \arrow[r, shift left, "{}^*g"] & \mathcal{D}_1 \arrow[l, shift left, "g"] \end{tikzcd}$$

$$\left.\begin{array}{l} \mathrm{Cone}\{{}^*g\circ g \to \mathrm{Id}\} \\ \mathrm{Cone}\{\mathrm{Id}\to g\circ {}^*g\} \end{array}\right\} \text{are equiv}$$

$N_\circ[g]$ categorifies $s_\circ(g)$ which relate $s_1$.

$g$ spherical $\Rightarrow$ perverse obj $S_\bullet(g)$

$\rightsquigarrow$ Defn of an $\infty$-cat of perverse Schobers of $(X,N)$
leads to

Schob$(X,N)$ = loc sys of $\infty$-cat on $X\setminus N$ w/ gluing data of spherical functors near $N$

$$\begin{tikzcd} & \mathrm{Schob}(X,N) \arrow[dl, "\mathcal{F}"'] \arrow[d, "K_0\otimes\mathbb{Q}"] \\ \mathrm{dgCat} \arrow[dr, "K_0\otimes\mathbb{Q}"'] & \mathrm{Perv}(X,N) \arrow[d, "\mathcal{F}\mapsto H^\circ(X{,}NX{;}\mathcal{F})"] \\ & \mathrm{Vect} \end{tikzcd}$$

top. Fuk. cat