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Basic notation

Base field $k = \bar{k}$

R gc (=graded connected) algebra if $R = \bigoplus_{i \geq 0} R_i$ $R_0 = k$ & f.gen as k-alg.

$\mathrm{gr}\, R = \mathbb{Z}$-graded noeth. R-modules

$\mathrm{qgr}(R) = \mathrm{gr}\, R / (\text{fin dim mod})$ "=" $\mathrm{coh}(\mathrm{Proj}\, R)$

Starting point Artin-Tate-Van den Bergh (ATV) - classification of non-comm. $\mathbb{P}^2$.

A.S. Gorenstein of dim d condition for R: means R fin inj dim

$\mathrm{Ext}^n_R(k, R) = \delta_{nd}\, k$

R is A.S. regular if gl dim $R = d$, GK dim $R < \infty$ & AS Gorenstein of dim d

Twisted Hom Coord Rings (TCR)

$$B(X, \mathcal{L}, \sigma) = \bigoplus H^0(X, \mathcal{L}_n) \qquad \mathcal{L}_n = \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \ldots \otimes (\sigma^*)^{n-1} \mathcal{L}$$

proj. scheme, line bundle, aut.

· Prop: (A-VdB) If $\mathcal{L}$ is σ-ample (in this lecture = ample)
then $\mathrm{qgr}(B) = \mathrm{coh}(X)$

· Thm (ATV, Stephenson): The AS reg. rings of dim 3 are classified
- they are all Noeth domains
- If Hilb series of R is $\frac{1}{(1-t)^3}$ then either:
  - $B(\mathbb{P}^2, \mathcal{O}(1), \sigma)$
  - $R \twoheadrightarrow B(E, \mathcal{L}, \sigma)$ (cubic curve)

generic case is E = elliptic & $|\sigma| = \infty \rightsquigarrow$ this is the Sklyanin algebra $S = Skl(E, \sigma)$. Here $B(E, \mathcal{L}, \sigma) = S/gS$, $S_3 \ni g$ central

Think of this classifying all $\mathbb{P}^2_{NC} = \mathrm{qgr}(R)$

Birational classifications: given $\bar{R}$ a domain of GKdim 3

$R \rightsquigarrow RC^{-1}$ $\qquad C = $ homog. elements

$\overset{\shortparallel}{D}[x, x^{-1}\sigma] = Q_{gr}(R)$

$D = D_{gr}(R) =$ "function field of R"

~~Hmm~~ Can we understand the alg R with $D_{gr}(R) = D_{gr}(Skl)$?

No

Take S, $T = S^{(3)} = \oplus S_{3n}$

Q: Find subrings of T

T is an elliptic algebra and domain, meaning $\exists$ a central element $g \in T$ and $T/gT \cong B(E, \mathcal{M}, \tau)$ $\quad |\tau| = \infty$

in our case $\mathcal{M} = \mathcal{L}_3$, $\tau = \sigma^3$

. Def: R be an degree $d \geq 2$ elliptic alg. D effective divisor on E

$\deg D \leq d-2$

$X = \{x \in R : \bar{x} = [x + gR] \in H^0(E, \mathcal{M} \otimes \mathcal{O}_E(-D))\}$

$\bar{x} \in H^0(\mathcal{M})$ vanishing on D

$\deg T = \deg \mathcal{M}$

$R \supset k\langle V \rangle = R(D) =$ "NC ~~blowup~~ blowup of R along D"

why? - It is $\cong$ VdB's blowup

- It has appropriate properties

If $D = D' + p$ (D' effective) $\quad R(D) = R(D')(p)$ so enough to consider $R(p)$

. $R(p)$ is elliptic $\quad R(p)/gR(p) \cong B(E, \mathcal{M}(-p), \tau)$ of degree $d - \deg(D)$

. $R/R(p) \cong \bigoplus_{n \geq 0} L[-n]$ $\quad$ here L is the "exceptional line module"

a line (module) is a module $L = L_0 R(p)$ with Hilb series $(1-t)^{-2}$ of $k[u,v]$

Think of this as blowing up the point module $\mathcal{M} = \mathcal{M}_p$ corresponding to the point $p \in E$.

The Point here is that E parametrises the point modules $\mathcal{M}$ of S (or T): - modules $\mathcal{M} = \mathcal{M}_0 S$ with H.S $\frac{1}{1-t}$ of $k[u]$

Thm • $T = S^{(3)}$ $\mathcal{U} = T(D)$, $\deg D \le 7$. Then $\mathcal{U}$ is a maximal order

i.e. if $\mathcal{U} \subseteq \mathcal{U}' \subseteq Q_{gr}(\mathcal{U})$ with $a\mathcal{U}'b \subseteq \mathcal{U}$ for $a, b \in \mathcal{U} \setminus 0$, then $\mathcal{U} = \mathcal{U}'$

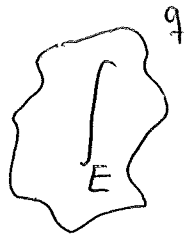
• $\mathcal{U}$ is nice - e.g. Noeth domain AS Gorenstein

Conversely if $V \subseteq T$ gen. in degree 1, with $Q_{gr}(T) = Q_{gr}(\mathcal{U})$ & a maximal order, then $\mathcal{U} = T(D)$ some D deg ≤ 7

Thm (RSS) Any max order $\mathcal{U} \subseteq T$ with $D_{gr}(\mathcal{U}) = D_{gr}(T)$
Then $\mathcal{U}$ is a Noeth domain and can be obtained from T by a more general form of blowing up (at "virtually effective" divisors)

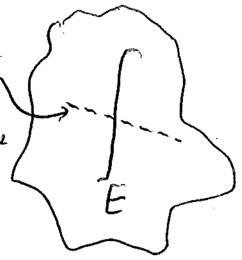
T

qgr T



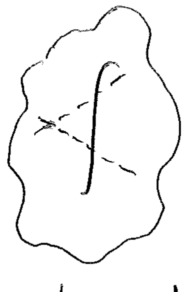
$T(p)$

one new line L with just one point p



no more points

$T(p + \tau^n p)$



with new fat point M
so now $\dim M_q = n$ $\forall n$?

$T(2p)$ - qgr(T) has ∞ hom dim but no "new points"

What about reversing this & blowing down line L (modules)? in $\mathcal{U}$

Self-intersection: $L \cdot L = \sum (-1)^{n+1} \dim \operatorname{Ext}^n_{\operatorname{qgr}\mathcal{U}}(L, L)$

Thm: If $\mathcal{U}$ is elliptic $\deg \mathcal{U} \ge 3$ & qgr$(\mathcal{U})$ has finite hom dim, then you can blow down any line L with $L \cdot L = -1$

Explicitly $\exists\, V \supseteq \mathcal{U}$ V elliptic s.t. $V/\mathcal{U} = \bigoplus_{n \ge 0} L[-n]$ & $\exists\, p \in E$ with $\mathcal{U} = V(p)$