Stellari | Bridgeland stab for semi⊥ decomp | jt w/ Bayer, Lahoz, Macri

Stay to example: Cubic 4 fold. Then indicate general strategy @ end.

$X \subset \mathbb{P}^5$ cubic$/\mathbb{C}$. Then $\underset{D^b Coh(X)}{\overset{\shortparallel}{D^b(X)}} = \langle \underbrace{Ku(X)}_{\text{"Kuznetsov component"}}, \underbrace{\mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)}_{\text{exceptional objects, } H \text{ hyperplane section}} \rangle$

Properties of $Ku(X)$

- $Ku(X)$ is 2-CY — $\underset{\text{Serre functor}}{S_{Ku(X)}} \cong [2]$.
- $HH_*(Ku(X)) \cong HH_*(K3 \text{ surface})$

ie, $Ku(X)$ is a "non-comm K3 surface."

Conj (Kuznetsov) $Ku(X)$ commutative (ie, $\exists$ K3 S st $D^b(S) \simeq Ku(X)$) iff X rational.

But we know when $Ku(X)$ is comm. Consider Noether-Lefschetz locus:

C_d divisor in moduli C of cubic 4-folds X s.t. $\exists$ primitive embedding of a pos def rank 2 lattice $K_d \hookrightarrow H^{2,2}(X,\mathbb{Z})$ of discriminant d, and containing H^2 ← self-intersection of hyperplane section.

Then Hassett + Addington-Thomas imply:

generically,

Ku(X) commutative iff

- $X \in C_d$
- $d \equiv 0, 2 \pmod 6$
- d not divis by 4, 9, or any prime p s.t. $p \neq 2$, $p \equiv 2 \pmod 3$.

$d = 0 \quad \overset{\checkmark}{2} \,|\, \overset{\checkmark}{6} \quad 8 \,|\, 12 \quad \overset{\checkmark}{14} \,|\, \cancel{18} \quad 20 \,|\, 24 \quad \overset{\checkmark}{26}$

Hybrechts says, generically ← twisted, "herby" version

$$\mathrm{Ku}(X) \simeq D^b(S, \alpha) \quad \alpha \in Br(S) = H^2(S, \mathcal{O}_X^*)_{Tor} \quad \text{iff } X \in C_d \text{ w/ } d \text{ s.t.}$$

- $d \equiv 0, 2 \pmod 6$
- $n_i \equiv 0 \pmod 2$ for all primes $p_i \equiv 2 \pmod 3$ in $2d = \prod p_i^{n_i}$.

So generically, Ku(X) is non-comm.

There are more similarities btw (twisted) K3 surfaces and Ku(X) of X cubic 4-fold.
(Wonder what "rational" becomes if Kuzn. conj. changed to "twisted K3.")

Recall
- (S_i, α_i) twisted K3 are FM partners iff $D^b(S_1, \alpha_1) \simeq D^b(S_2, \alpha_2)$

  Fourier-Mukai   Defn

- Huybrechts: ~~(twisted) FM~~ X_1, X_2 FM partners iff $\exists$ exact equiv $\mathrm{Ku}(X_1) \simeq \mathrm{Ku}(X_2)$ s.t. $D^b(X_1) \to \mathrm{Ku}(X_1) \simeq \mathrm{Ku}(X_2) \hookrightarrow D^b(X_2)$ is ~~FM type~~. a FM functor.

③

Twisted K3	Cubic 4-folds (Huybrechts)
(1) (S_i, α_i) FM partners $\Updownarrow$ $\exists\ \tilde{H}(S_1,\alpha_1,\mathbb{Z}) \cong \tilde{H}(S_2,\alpha_2,\mathbb{Z})$ orient. preserving Hodge isometry. "Mukai ~~pairing~~ pairing" includes $H^0, H^2, H^4 \ldots$ Hodge structure as $H^{2,0}$ dim 1, $H^{0,2}$ dim 1. 4 positive dims on lattice; must preserve positivity	(1) X_1, X_2 FM partners $\Updownarrow$ $\exists$ orient-pres. Hodge isom. $\tilde{H}(Ku(X_1),\mathbb{Z}) \cong \tilde{H}(Ku(X_2),\mathbb{Z})$
(2) Up to isom, (S_1,α_1) has only finitely many FM partners (Bridgeland-Maciocia, Huybrechts-S)	(2) Likewise for X cubic 4-fold.
(3) # of FM partners of (S,α) can be arbitrarily large. (Oguiso, Ma)	(3) Likewise.

$\boxed{\text{all these results based on geom of moduli spaces of stable sheaves on K3 surfaces}}$

(use Bridgeland stab. to study birational geom of these moduli)

If (S,α) twisted K3, the space

$$\mathrm{Stab}(S,\alpha) \neq \emptyset$$

and can describe a nice conn. component in $\mathrm{Stab}(S,\alpha)$

$$\Downarrow$$

If X cubic 4fold of "geometric nature," ($\mathrm{Ku}(X) \simeq D^b(S,\alpha)$)
then $\mathrm{Stab}(\mathrm{Ku}(X)) \neq \emptyset$.

(Recall given $\mathcal{T}$ triang., $v: K_0(\mathcal{T}) \twoheadrightarrow \Lambda$ surjective

finite rank lattice

a stab cond is $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A}$ heart of bdd t-str on $\mathcal{T}$,
$Z: \Lambda \to \mathbb{C}$ grp homom s.t.

(1) $Z(v(E)) \in$ 

$\forall\, E \neq 0,\ E \in \mathcal{A}$

(2) Harder-Narasimhan

(3) support property.

where Λ comes up!

$\Rightarrow$ wall+chamber structure on Stab.)

Question: Is $\mathrm{Stab}(\mathrm{Ku}(X)) \neq \emptyset$?

Application: (1) Study birat. geom of:
- Fano var. of lines on X
- "rational normal curve" moduli on X

(2) Remove genericity assump. on AT, Hyperk.

<u>Thm</u> (BLMS) X any cubic 4fold, then $\mathrm{Stab}(\mathrm{Ku}(X)) \neq \emptyset$ (Can describe Λ, but not today.)