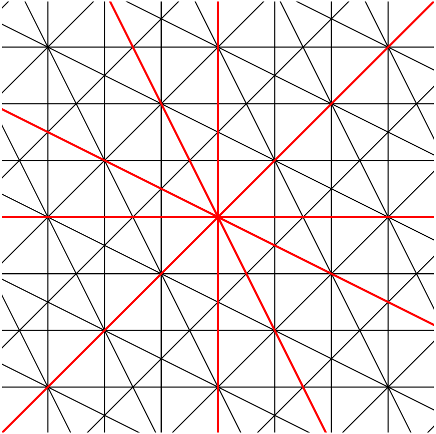
Affine actions from 3-fold flops, and tilings of the plane

Michael Wemyss

University of Glasgow
www.maths.gla.ac.uk/∼mwemyss

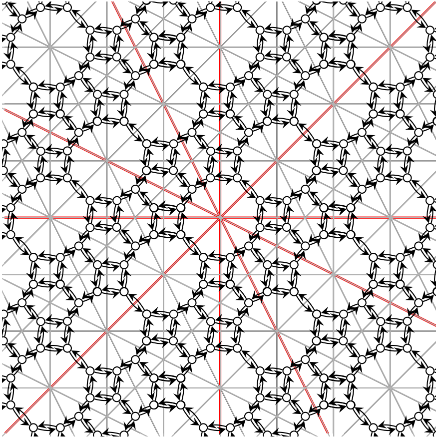
Purpose of this Talk

To explain the following tiling, which I view as an affine version of symmetries of the pentagon D_5, and its applications.



Purpose of this Talk

To explain the following tiling, which I view as an affine version of symmetries of the pentagon D_5, and its applications.



Motivation I: Surfaces

Suppose that $f\colon X \to \mathbb{C}^2/G$ is the minimal resolution of a Kleinian singularity, that is, $G \leq \mathrm{SL}(2,\mathbb{C})$. The key point is that the fibre $C := f^{-1}(0)$ with reduced scheme structure decomposes

$$C^{\mathrm{red}} = \bigcup_{i=1}^{n} C_i$$

with each $C_i \cong \mathbb{P}^1$. The curves intersect in an ADE arrangement.

Motivation I: Surfaces

Suppose that $f\colon X \to \mathbb{C}^2/G$ is the minimal resolution of a Kleinian singularity, that is, $G \leq \mathrm{SL}(2,\mathbb{C})$. The key point is that the fibre $C := f^{-1}(0)$ with reduced scheme structure decomposes

$$C^{\mathrm{red}} = \bigcup_{i=1}^{n} C_i$$

with each $C_i \cong \mathbb{P}^1$. The curves intersect in an ADE arrangement.

The sheaves $E_i := \mathcal{O}_{C_i}(-1)$, and also the sheaf $E_f := \mathcal{O}_C$, are examples of spherical objects, namely

$$E \otimes \omega_X \cong E \quad \text{and} \quad \mathrm{Ext}^t_X(E,E) \cong \begin{cases} \mathbb{C} & \text{if } t = 0, 2 \\ 0 & \text{else.} \end{cases}$$

Motivation II

Recall that, for an ADE (or extended ADE) graph Γ, the *braid group* is defined

$$B_\Gamma := \langle s_i \mid s_i s_j s_i = s_j s_i s_j \text{ or } s_i s_j = s_j s_i \rangle$$

Seidel–Thomas: there is an action of the braid group on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, that is, there is a group homomorphism

$$B_\Gamma \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

which sends $s_i \mapsto T_{E_i}$.

Motivation II

Recall that, for an ADE (or extended ADE) graph Γ, the *braid group* is defined

$$B_\Gamma := \langle s_i \mid s_i s_j s_i = s_j s_i s_j \text{ or } s_i s_j = s_j s_i \rangle$$

Seidel–Thomas: there is an action of the braid group on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, that is, there is a group homomorphism

$$B_\Gamma \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

which sends $s_i \mapsto T_{E_i}$.

Brav–Thomas: the above homomorphism is injective.

Motivation II

Recall that, for an ADE (or extended ADE) graph Γ, the *braid group* is defined

$$B_\Gamma := \langle s_i \mid s_i s_j s_i = s_j s_i s_j \text{ or } s_i s_j = s_j s_i \rangle$$

Seidel–Thomas: there is an action of the braid group on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, that is, there is a group homomorphism

$$B_\Gamma \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

which sends $s_i \mapsto T_{E_i}$.

Brav–Thomas: the above homomorphism is injective.

By also throwing in the twist of E_f, there is an *affine action*

$$B_{\widetilde{\Gamma}} \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

A 3-dimensional version?

The question, motivated both by birational geometry and homological algebra, is how to lift this to dimension three.

A 3-dimensional version?

The question, motivated both by birational geometry and homological algebra, is how to lift this to dimension three.

A natural setting in which is ask the question is for a 3-fold flopping contraction $f\colon X \to X_{\mathrm{con}}$. With this, the question is:

Dim	Use Curves $E_1, \ldots, E_n$	Add in E_f
2	$B_\Gamma \to \mathrm{D^b}(\mathrm{coh}\, X)$	$B_{\widetilde{\Gamma}} \to \mathrm{D^b}(\mathrm{coh}\, X)$
3	?	??

What is a flop?

Answer 1: A birational surgery.

What is a flop?

Answer 1: A birational surgery.

Definition

Suppose that $f\colon X \to X_{\text{con}}$ is a crepant projective birational morphism, contracting a curve C to a point p, such that f is an isomorphism away from C.

$$X \searrow X_{\text{con}}$$

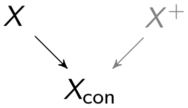
Then we say that $f^{+}\colon X^{+} \to X_{\text{con}}$ is the flop of f if for every line bundle $\mathcal{L} = \mathcal{O}_X(D)$ on X such that $-D$ is f-nef, then the proper transform of D is Cartier, and f^{+}-nef.

What is a flop?

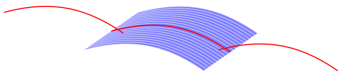
Answer 1: A birational surgery.

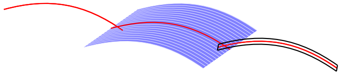
Definition

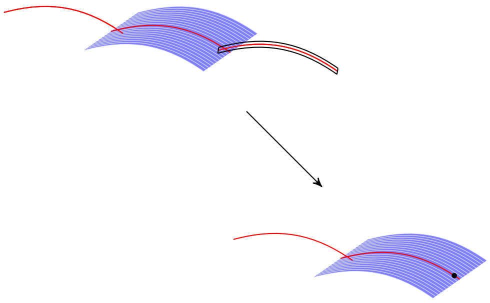
Suppose that $f\colon X \to X_{\text{con}}$ is a crepant projective birational morphism, contracting a curve C to a point p, such that f is an isomorphism away from C.

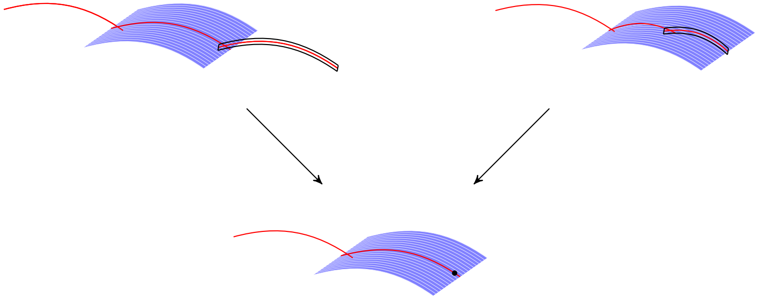


Then we say that $f^+\colon X^+ \to X_{\text{con}}$ is the flop of f if for every line bundle $\mathcal{L} = \mathcal{O}_X(D)$ on X such that $-D$ is f-nef, then the proper transform of D is Cartier, and f^+-nef.









What is a flop?

Answer 2: (Homological MMP) Its the solution to a universal problem in homological algebra, a *mutation* on the base X_{con}.

What is a flop?

Answer 2: (Homological MMP) Its the solution to a universal problem in homological algebra, a *mutation* on the base X_{con}.

The Homological MMP gives us a way, via cluster theory, to produce the flop in a manner suitable for iterations.

What is a flop?

Answer 2: (Homological MMP) Its the solution to a universal problem in homological algebra, a *mutation* on the base X_{con}.

The Homological MMP gives us a way, via cluster theory, to produce the flop in a manner suitable for iterations.

Rest of talk: we thus let $X \to X_{\text{con}}$ be a 3-fold flopping contraction, and we will construct affine and non-affine actions. Features:

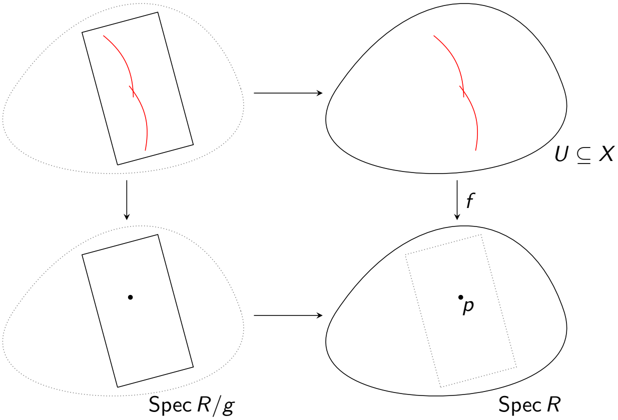
- The fibre above the origin need not be ADE any more!
- The case of 2 curves above the origin is not just A_2.
- Objects no longer spherical: need to deform. The fibre twist is particularly technical.

From Flops to Shaded Dynkin Diagrams: The Elephant

To understand smooth 3-folds X, we are forced to understand autoequivalences on singular surfaces. As a consequence, X might as well be singular too.

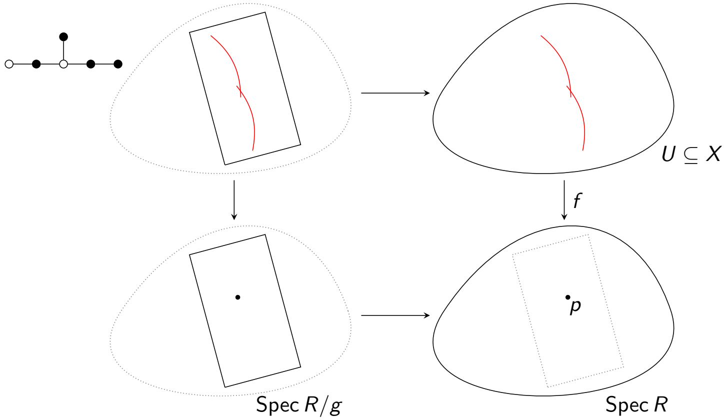
From Flops to Shaded Dynkin Diagrams: The Elephant

To understand smooth 3-folds X, we are forced to understand autoequivalences on singular surfaces. As a consequence, X might as well be singular too.



From Flops to Shaded Dynkin Diagrams: The Elephant

To understand smooth 3-folds X, we are forced to understand autoequivalences on singular surfaces. As a consequence, X might as well be singular too.

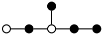


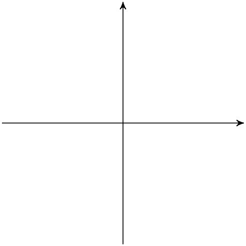
Intersection Combinatorics I

From a shaded Dynkin diagram, we produce both affine and non-affine group actions.

Intersection Combinatorics I

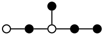
From a shaded Dynkin diagram, we produce both affine and non-affine group actions.

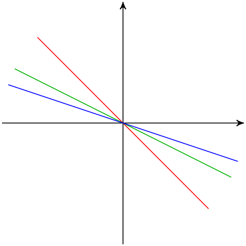
Non-affine version: A finite simplicial hyperplane arrangement is obtained as follows. In the example , the two white dots give 2 roots in the E_6 root system, and hence span a plane. We then *intersect* this plane with all the remaining reflection hyperplanes:



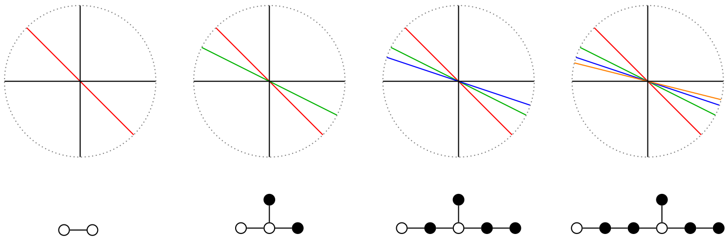
Intersection Combinatorics I

From a shaded Dynkin diagram, we produce both affine and non-affine group actions.

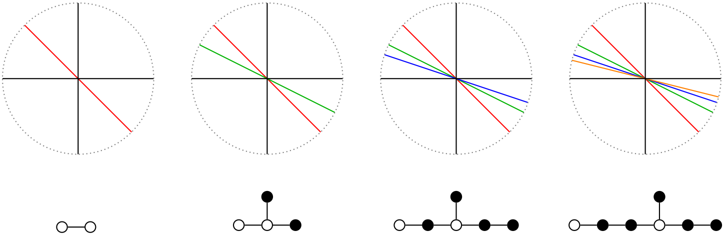
Non-affine version: A finite simplicial hyperplane arrangement is obtained as follows. In the example , the two white dots give 2 roots in the E_6 root system, and hence span a plane. We then *intersect* this plane with all the remaining reflection hyperplanes:

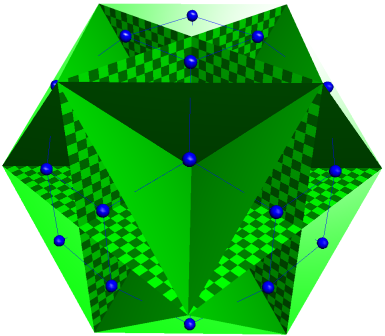


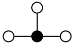
Some other examples:



Some other examples:





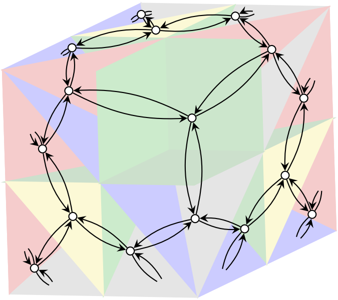


The Deligne Groupoid

Associated to every simplicial hyperplane arrangement is the *Deligne groupoid*

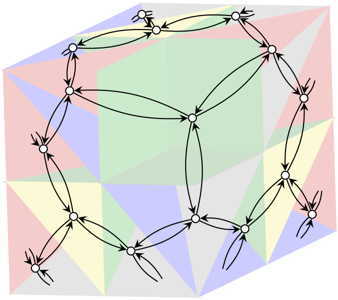
The Deligne Groupoid

Associated to every simplicial hyperplane arrangement is the *Deligne groupoid*



The Deligne Groupoid

Associated to every simplicial hyperplane arrangement is the *Deligne groupoid*



subject to the relations that identify minimal paths. It is well-known that any vertex group of this groupoid is isomorphic to the fundamental group of the complexified complement of the real hyperplane arrangement, and so we denote it $\pi_1(\mathbb{G})$.

The Non-Affine Result

Theorem

Suppose that $X \to X_{\text{con}}$ is a 3-fold flopping contraction, where X is reasonable (e.g. smooth), and each of the curves is individually floppable. Then:

1. *(Donovan–W) There is a group homomorphism*

$$\rho \colon \pi_1(\mathbb{G}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

   obtained by composing flop functors.

The Non-Affine Result

Theorem

Suppose that $X \to X_{\text{con}}$ is a 3-fold flopping contraction, where X is reasonable (e.g. smooth), and each of the curves is individually floppable. Then:

1. *(Donovan–W) There is a group homomorphism*

$$\rho\colon \pi_1(\mathbb{G}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

   obtained by composing flop functors.

2. *(Hirano–W) ρ is injective, that is, the action is faithful.*

The Non-Affine Result

Theorem

Suppose that $X \to X_{\text{con}}$ is a 3-fold flopping contraction, where X is reasonable (e.g. smooth), and each of the curves is individually floppable. Then:

1. *(Donovan–W) There is a group homomorphism*

$$\rho\colon \pi_1(\mathbb{G}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

   obtained by composing flop functors.

2. *(Hirano–W) ρ is injective, that is, the action is faithful.*

- The proofs are quite different from Seidel–Thomas and Brav–Thomas. We don't know or use generators and relations of $\pi_1(\mathbb{G})$, and there is no 'formula' for the flop functor.
- We view $\pi_1(\mathbb{G})$ as a *pure braid group*.

Intersection Combinatorics II

Given a shaded Dynkin diagram, we produce an affine version by playing the same intersection trick, but this time inside the *Tits cone*, instead of inside the usual root system. Recall that

$$Tits = \bigcup_{w \in \widetilde{W}} w(C_+).$$

Intersection Combinatorics II

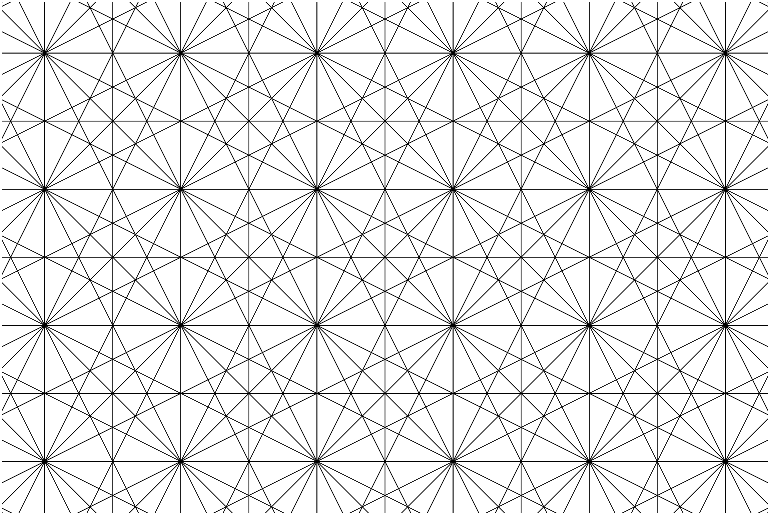
Given a shaded Dynkin diagram, we produce an affine version by playing the same intersection trick, but this time inside the *Tits cone*, instead of inside the usual root system. Recall that

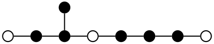
$$Tits = \bigcup_{w \in \widetilde{W}} w(C_+).$$

- This is harder to draw! It takes quite a bit of effort, and new combinatorics, to describe the intersection.
- Upshot: when the number of nodes equals two, we obtain a tiling of the plane.

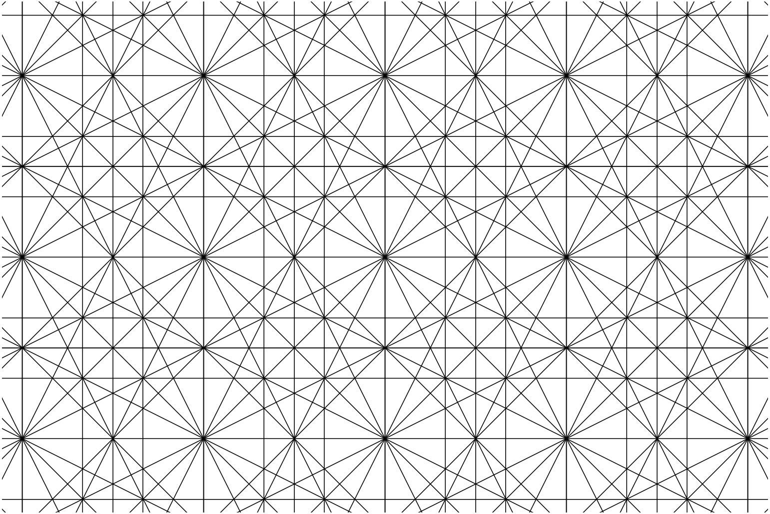
Remarkably, the tilings produced are new.

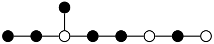
Some Examples





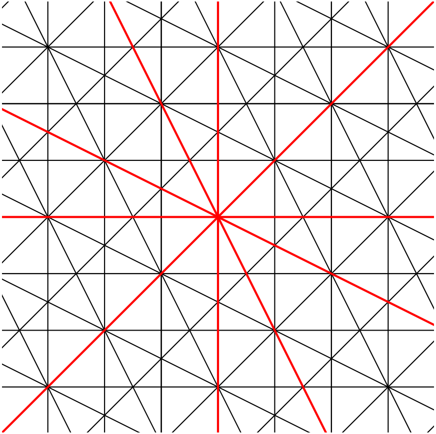
Some Examples





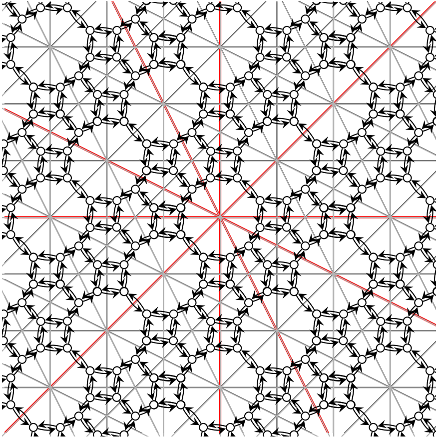
The Arrangement Groupoid

Now, form an infinite groupoid $\mathbb{G}_{\text{aff}}$ in the natural way



The Arrangement Groupoid

Now, form an infinite groupoid $\mathbb{G}_{\mathrm{aff}}$ in the natural way



subject to the tile relations. Write $\pi_1(\mathbb{G}_{\mathrm{aff}})$ for a vertex group of this groupoid.

The Main Result

Theorem (Iyama–W)

Suppose that $X \to X_{\mathrm{con}}$ is a 3-fold flopping contraction, where X is reasonable (e.g. smooth), and each of the curves is individually floppable. Then there is a group homomorphism

$$\widetilde{\rho}\colon \pi_1(\mathbb{G}_{\mathrm{aff}}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

The Main Result

Theorem (Iyama–W)

Suppose that $X \to X_{\text{con}}$ is a 3-fold flopping contraction, where X is reasonable (e.g. smooth), and each of the curves is individually floppable. Then there is a group homomorphism

$$\widetilde{\rho} \colon \pi_1(\mathbb{G}_{\text{aff}}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

- We view $\pi_1(\mathbb{G}_{\text{aff}})$ as some kind of analogue of an *affine pure braid group*, and we refer to the above as the *affine action*.
- The tilings of the plane above are only the baby case where there are two flopping curves! The theorem also deals with the case when there are more curves: these give 'tilings' of $\mathbb{R}^d$.